

# INTRINSIC ULTRA CONTRACTIVITY OF FEYNMAN-KAC SEMIGROUPS FOR SYMMETRIC JUMP PROCESSES

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ABSTRACT. Consider a symmetric non-local Dirichlet form  $(D, \mathcal{D}(D))$  given by

$$D(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y) dx dy$$

with  $\mathcal{D}(D)$  the closure of the set of  $C^1$  functions on  $\mathbb{R}^d$  with compact support under the norm  $\sqrt{D_1(f, f)}$ , where  $D_1(f, f) := D(f, f) + \int f^2(x) dx$  and  $J(x, y)$  is a nonnegative symmetric measurable function on  $\mathbb{R}^d \times \mathbb{R}^d$ . Suppose that there is a Hunt process  $(X_t)_{t \geq 0}$  on  $\mathbb{R}^d$  corresponding to  $(D, \mathcal{D}(D))$ , and that  $(L, \mathcal{D}(L))$  is its infinitesimal generator. We study the intrinsic ultracontractivity for the Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  generated by  $L^V := L - V$ , where  $V \geq 0$  is a non-negative locally bounded measurable function such that Lebesgue measure of the set  $\{x \in \mathbb{R}^d : V(x) \leq r\}$  is finite for every  $r > 0$ . By using intrinsic super Poincaré inequalities and establishing an explicit lower bound estimate for the ground state, we present general criteria for the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$ . In particular, if

$$J(x, y) \asymp |x - y|^{-d-\alpha} \mathbf{1}_{\{|x-y| \leq 1\}} + e^{-|x-y|^\gamma} \mathbf{1}_{\{|x-y| > 1\}}$$

for some  $\alpha \in (0, 2)$  and  $\gamma \in (1, \infty]$ , and the potential function  $V(x) = |x|^\theta$  for some  $\theta > 0$ , then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if and only if  $\theta > 1$ . When  $\theta > 1$ , we have the following explicit estimates for the ground state  $\phi_1$

$$c_1 \exp\left(-c_2 \theta^{\frac{\gamma-1}{\gamma}} |x| \log^{\frac{\gamma-1}{\gamma}}(1+|x|)\right) \leq \phi_1(x) \leq c_3 \exp\left(-c_4 \theta^{\frac{\gamma-1}{\gamma}} |x| \log^{\frac{\gamma-1}{\gamma}}(1+|x|)\right),$$

where  $c_i > 0$  ( $i = 1, 2, 3, 4$ ) are constants. We stress that our method efficiently applies to the Hunt process  $(X_t)_{t \geq 0}$  with finite range jumps, and some irregular potential function  $V$  such that  $\lim_{|x| \rightarrow \infty} V(x) \neq \infty$ .

**Keywords:** symmetric jump process; Dirichlet form; intrinsic ultracontractivity; Feynman-Kac semigroup; compactness; super Poincaré inequality; Lévy process

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## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Setting and assumptions.** Let  $(D, \mathcal{D}(D))$  be a symmetric non-local Dirichlet form as follows

$$(1.1) \quad \begin{aligned} D(f, f) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y) dx dy, \\ \mathcal{D}(D) &= \overline{C_c^1(\mathbb{R}^d)}^{D_1}, \end{aligned}$$

where  $J(x, y)$  is a non-negative measurable function on  $\mathbb{R}^d \times \mathbb{R}^d$  satisfying that

$$(1) \quad J(x, y) = J(y, x) \text{ for all } x, y \in \mathbb{R}^d;$$

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(2) There exist  $\alpha_1, \alpha_2 \in (0, 2)$  with  $\alpha_1 \leq \alpha_2$  and positive  $\kappa, c_1, c_2$  such that

$$(1.2) \quad c_1|x-y|^{-d-\alpha_1} \leq J(x, y) \leq c_2|x-y|^{-d-\alpha_2}, \quad 0 < |x-y| \leq \kappa$$

and

$$(1.3) \quad \sup_{x \in \mathbb{R}^d} \int_{\{|x-y| > \kappa\}} J(x, y) dy < \infty.$$

Here,  $C_c^1(\mathbb{R}^d)$  denotes the space of  $C^1$  functions on  $\mathbb{R}^d$  with compact support,  $D_1(f, f) := D(f, f) + \int f^2(x) dx$  and  $\mathcal{D}(D)$  is the closure of  $C_c^1(\mathbb{R}^d)$  with respect to the metric  $D_1(f, f)^{1/2}$ . It is easy to see that (1.2) and (1.3) imply that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |x-y|^2) J(x, y) dy < \infty,$$

which in turn gives us that  $D(f, f) < \infty$  for each  $f \in C_c^1(\mathbb{R}^d)$ . According to [12, Example 1.2.4], we know that  $(D, \mathcal{D}(D))$  is a regular Dirichlet form on  $L^2(\mathbb{R}^d; dx)$ . Therefore, there exists  $N \subset \mathbb{R}^d$  having zero capacity with respect to the Dirichlet form  $(D, \mathcal{D}(D))$ , and there is a Hunt process  $((X_t)_{t \geq 0}, \mathbb{P}^x)$  with state space  $\mathbb{R}^d \setminus N$  such that for every  $f \in L^2(\mathbb{R}^d; dx)$  and  $t > 0$ ,  $x \mapsto \mathbb{E}^x(f(X_t))$  is a quasi-continuous version of  $T_t f$ , where  $\mathbb{E}^x$  is the expectation under the probability measure  $\mathbb{P}^x$  and  $(T_t)_{t \geq 0}$  is the  $L^2$ -semigroup associated with  $(D, \mathcal{D}(D))$ , see e.g. [1, Theorem 1.1]. The set  $N$  is called the properly exceptional set of the process  $(X_t)_{t \geq 0}$  (or, equivalently, of the Dirichlet form  $(D, \mathcal{D}(D))$ ), and it has zero Lebesgue measure. Furthermore, by (1.2), (1.3) and the proof of [1, Theorem 1.2], there exists a positive symmetric measurable function  $p(t, x, y)$  defined on  $[0, \infty) \times (\mathbb{R}^d \setminus N) \times (\mathbb{R}^d \setminus N)$  such that

$$T_f(x) = \mathbb{E}^x(f(X_t)) = \int_{\mathbb{R}^d \setminus N} p(t, x, y) f(y) dy, \quad x \in \mathbb{R}^d \setminus N, \quad t > 0, \quad f \in B_b(\mathbb{R}^d);$$

moreover, for every  $t > 0$  and  $y \in \mathbb{R}^d \setminus N$ , the function  $x \mapsto p(t, x, y)$  is quasi-continuous on  $\mathbb{R}^d \setminus N$ , and for any  $t > 0$  there is a constant  $c_t > 0$  such that for any  $x, y \in \mathbb{R}^d \setminus N$ ,  $0 < p(t, x, y) \leq c_t$ .

First, we make the following continuity assumption on  $p(t, x, y)$ .

**(A1)**  $N = \emptyset$ . For every  $t > 0$ , the function  $(x, y) \mapsto p(t, x, y)$  is continuous on  $\mathbb{R}^d \times \mathbb{R}^d$ , and  $0 < p(t, x, y) \leq c_t$  for all  $x, y \in \mathbb{R}^d$ .

In particular, **(A1)** implies that the Hunt process  $((X_t)_{t \geq 0}, \mathbb{P}^x)$  is well defined for all  $x \in \mathbb{R}^d$ , and the associated strongly continuous Markov semigroup  $(T_t)_{t \geq 0}$  is ultracontractive, i.e.  $\|T_t f\|_{L^\infty(\mathbb{R}^d; dx)} \leq c_t \|f\|_{L^1(\mathbb{R}^d; dx)}$  for all  $t > 0$  and every  $f \in L^1(\mathbb{R}^d; dx)$ .

When for any  $x, y \in \mathbb{R}^d$ ,  $J(x, y) = \rho(x - y)$  holds with some non-negative measurable function  $\rho$  on  $\mathbb{R}^d$  such that  $\rho(z) = \rho(-z)$  for all  $z \in \mathbb{R}^d$  and  $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \rho(z) dz < \infty$ , the corresponding Hunt process  $(X_t)_{t \geq 0}$  is a symmetric Lévy process having Lévy jump measure  $\nu(dz) := \rho(z) dz$ . In this case, assumption **(A1)** is equivalent to  $e^{-t\Psi_0(\cdot)} \in L^1(\mathbb{R}^d; dx)$  for any  $t > 0$ , where the characteristic exponent or the symbol  $\Psi_0$  of Lévy process  $(X_t)_{t \geq 0}$  is defined by

$$\mathbb{E}^x(e^{i\langle \xi, X_t - x \rangle}) = e^{-t\Psi_0(\xi)}, \quad x, \xi \in \mathbb{R}^d, \quad t > 0.$$

It is well known that the Lévy process enjoys the space-homogeneous property. For sufficient conditions on the jump density  $J(x, y)$  such that the associated space-inhomogeneous Hunt process  $(X_t)_{t \geq 0}$  satisfies assumption **(A1)**, we refer the reader to [4, 5, 6, 1, 7] and the references therein.

Let  $V$  be a non-negative measurable and locally bounded potential function on  $\mathbb{R}^d$ . Define the Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  associated with the Hunt process  $(X_t)_{t \geq 0}$  as follows:

$$T_t^V(f)(x) = \mathbb{E}^x \left( \exp \left( - \int_0^t V(X_s) ds \right) f(X_t) \right), \quad x \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d; dx).$$

It is easy to check that  $(T_t^V)_{t \geq 0}$  is a bounded symmetric semigroup on  $L^2(\mathbb{R}^d; dx)$ . Furthermore, following the arguments in [10, Section 3.2] (see also the proof of [15, Lemma 3.1]), we can find that for each  $t > 0$ ,  $T_t^V$  is a bounded operator from  $L^1(\mathbb{R}^d; dx)$  to  $L^\infty(\mathbb{R}^d; dx)$ , and there exists a bounded, positive and symmetric transition kernel  $p^V(t, x, y)$  on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  such that for any  $t > 0$ , the function  $(x, y) \mapsto p^V(t, x, y)$  is continuous on  $\mathbb{R}^d \times \mathbb{R}^d$ , and for every  $1 \leq p \leq \infty$ ,

$$T_t^V f(x) = \int_{\mathbb{R}^d} p^V(t, x, y) f(y) dy, \quad x \in \mathbb{R}^d, f \in L^p(\mathbb{R}^d; dx).$$

The following result gives us an easy criterion for the compactness of the semigroup  $(T_t^V)_{t \geq 0}$ . The proof is mainly based on [21, Corollary 1.3]. For the sake of completeness, we will provide its proof in the Appendix.

**Proposition 1.1.** *Under Assumption **(A1)**, if for any  $r > 0$ , Lebesgue measure of the set*

$$\{x \in \mathbb{R}^d : V(x) \leq r\}$$

*is finite, then the semigroup  $(T_t^V)_{t \geq 0}$  is compact.*

From now on, we will take the following assumption:

**(A2)** *Lebesgue measure of the set  $\{x \in \mathbb{R}^d : V(x) \leq r\}$  is finite for any  $r > 0$ .*

In particular, according to Proposition 1.1, the semigroup  $(T_t^V)_{t \geq 0}$  is compact. By general theory of semigroups for compact operators, there exists an orthonormal basis of eigenfunctions  $\{\phi_n\}_{n=1}^\infty$  in  $L^2(\mathbb{R}^d; dx)$  associated with corresponding eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  satisfying  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  and  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . That is,  $L^V \phi_n = -\lambda_n \phi_n$  and  $T_t^V \phi_n = e^{-\lambda_n t} \phi_n$ , where  $(L^V, \mathcal{D}(L^V))$  denotes the infinitesimal generator of the semigroup  $(T_t^V)_{t \geq 0}$ . The first eigenfunction  $\phi_1$  is called ground state in the literature. Furthermore, according to assumptions above, we have the following property for  $\phi_1$ . The proof is also left to the Appendix.

**Proposition 1.2.** *Under Assumptions **(A1)** and **(A2)**, there exists a version of  $\phi_1$  which is bounded, continuous and strictly positive.*

To derive a upper bound estimate for the ground state  $\phi_1$ , we need the explicit expression of the operator  $L^V$ , which is given by

$$L^V f(x) = Lf(x) - V(x)f(x).$$

Here,  $(L, \mathcal{D}(L))$  is the generator associated with Dirichlet form  $(D, \mathcal{D}(D))$ . In Lévy case, it is easy to see that for any  $f \in C_c^2(\mathbb{R}^d) \subset \mathcal{D}(L)$ ,

$$Lf(x) = \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \rho(z) dz,$$

where  $\rho$  is the density function of the Lévy measure. For general non-local Dirichlet form  $(D, \mathcal{D}(D))$ , if for every  $x \in \mathbb{R}^d$ ,

$$\int_{\{|z| \leq 1\}} |z| |J(x, x+z) - J(x, x-z)| dz < \infty,$$

and for any  $r > 0$  large enough,

$$x \mapsto \mathbf{1}_{B(0, 2r)^c} \int_{\{|x+z| \leq r\}} J(x, x+z) dz \in L^2(\mathbb{R}^d; dx),$$

then  $C_c^2(\mathbb{R}^d) \subset \mathcal{D}(L)$  and for any  $f \in C_c^2(\mathbb{R}^d)$ ,

$$\begin{aligned} Lf(x) &= \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) J(x, x+z) dz \\ &\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle (J(x, x+z) - J(x, x-z)) dz, \end{aligned}$$

e.g. see [22, Theorem 1.1] for more details. According to the discussions above, sometime we adopt the following regular assumptions on  $J(x, y)$  and the operator  $L^V$ , which are satisfied for all symmetric Lévy processes.

**(A3)** *The jump kernel  $J(x, y)$  satisfies that*

$$\sup_{x \in \mathbb{R}^d} \int_{\{|z| \leq 1\}} |z| |J(x, x+z) - J(x, x-z)| dz < \infty,$$

*and for any  $f \in C_c^2(\mathbb{R}^d) \subseteq \mathcal{D}(L^V)$ ,*

$$\begin{aligned} (1.4) \quad L^V f(x) &= \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) J(x, x+z) dz \\ &\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle (J(x, x+z) - J(x, x-z)) dz - V(x)f(x). \end{aligned}$$

**1.2. Main results.** *Throughout this paper, we always assume that assumptions (A1) and (A2) hold, and that the ground state  $\phi_1$  is bounded, continuous and strictly positive.* In this paper, we are concerned with the intrinsic ultracontractivity for the semigroup  $(T_t^V)_{t \geq 0}$ . We first recall the definition of intrinsic ultracontractivity for Feynman-Kac semigroups introduced in [11]. The semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if and only if for any  $t > 0$ , there exists a constant  $C_t > 0$  such that for all  $x, y \in \mathbb{R}^d$ ,

$$p^V(t, x, y) \leq C_t \phi_1(x) \phi_1(y).$$

In the framework of the semigroup theory, define

$$(1.5) \quad \tilde{T}_t^V f(x) := \frac{e^{\lambda_1 t}}{\phi_1(x)} T_t^V((\phi_1 f))(x), \quad t > 0,$$

which is a Markov semigroup on  $L^2(\mathbb{R}^d; \phi_1^2(x) dx)$ . Then,  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if and only if  $(\tilde{T}_t^V)_{t \geq 0}$  ultracontractive, i.e., for every  $t > 0$ ,  $\tilde{T}_t^V$  is a bounded operator from  $L^2(\mathbb{R}^d; \phi_1^2(x) dx)$  to  $L^\infty(\mathbb{R}^d; \phi_1^2(x) dx)$ .

Recently, the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  associated with some special pure jump symmetric Lévy process  $(X_t)_{t \geq 0}$  has been investigated in [15, 13, 14]. The approach of all these cited papers is based on sharp and explicit pointwise upper and lower bound estimates for the ground state  $\phi_1$  corresponding to the semigroup  $(T_t^V)_{t \geq 0}$ . However, to apply such powerful technique, some restrictions on the density function of jump kernel are needed, e.g. see [14, Assumption 2.1]. In particular, in Lévy case the following typical example

$$(1.6) \quad J(x, y) \asymp |x - y|^{-d-\alpha} \mathbb{1}_{\{|x-y| \leq 1\}} + e^{-|x-y|^\gamma} \mathbb{1}_{\{|x-y| > 1\}}$$

with  $\alpha \in (0, 2)$  and  $\gamma \in (1, \infty]$  is not included in [15, 14, 13]. Here and in what follows, for two functions  $f$  and  $g$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$ ,  $f \asymp g$  means that there is a constant  $c > 1$  such that  $c^{-1}g(x, y) \leq f(x, y) \leq cg(x, y)$  for all  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ . In particular, when  $\gamma = \infty$ ,

$$J(x, y) \asymp |x - y|^{-d-\alpha} \mathbb{1}_{\{|x-y| \leq 1\}},$$

which is associated with the truncated symmetric  $\alpha$ -stable process. As mentioned in [6, 7, 1], such jump density function  $J(x, y)$  is very important in applications, and it arises in statistical physics to model turbulence as well as in mathematical finance to model stochastic volatility.

Furthermore, the following growth condition on the potential function

$$(1.7) \quad \lim_{|x| \rightarrow \infty} V(x) = \infty$$

was commonly used in [15, 13, 14] to derive the compactness of  $(T_t^V)_{t \geq 0}$ , e.g. [14, Assumption 2.4]. However, as shown by Proposition 1.2, assumption **(A2)**, which is much weaker than (1.7), is sufficient to ensure the compactness of  $(T_t^V)_{t \geq 0}$ . Therefore, a natural question is whether one can give some sufficient conditions for the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  without the restrictive condition (1.7).

In this paper, we will make use of super Poincaré inequalities with respect to infinite measure developed in [18] and functional inequalities for non-local Dirichlet forms recently studied in [20, 23, 3] to deal with the questions mentioned above. We aim to present some sharp conditions on the potential function  $V$  such that the associated Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive, and also derive explicit two-sided estimates for the ground state  $\phi_1$ . Our method is different from that of [15, 13, 14], and deals with the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  for non-local Dirichlet forms in more general situations. The following points indicate the novelties of our paper.

- (i) We can deal with the example  $J(x, y)$  mentioned in (1.6), which essentially means that small jumps play the dominant roles for the behavior of the associated process. On the other hand, by (1.2), we also consider the case that the density of the small jumps can enjoy the variable order property.
- (ii) For a large class of potential functions  $V$  which do not satisfy the growth condition (1.7) or the regularity condition (1.11) below, we can still obtain some sufficient conditions for the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$ , which to the best of our knowledge do not appear in the literature.
- (iii) Our method here efficiently applies to Hunt process generated by non-local Dirichlet forms. In particular, the associated process does not like Lévy process, and it is usually not space-homogeneous.

Now, we will present main results of our paper, which will be split into two subsections.

**1.2.1. The case that  $\lim_{|x| \rightarrow \infty} V(x) = \infty$ .** The following statement is a consequence of more general Theorem 3.8 below.

**Theorem 1.3.** *Suppose that (1.2), (1.3), (A1) and (A2) hold, and that there exist positive constants  $c_i$  ( $i = 3, 4$ ),  $\theta_i$  ( $i = 1, 3$ ) and constants  $\theta_i$  ( $i = 2, 4$ ) such that for every  $x \in \mathbb{R}^d$  with  $|x|$  large enough,*

$$(1.8) \quad c_3|x|^{\theta_1} \log^{\theta_2}(1 + |x|) \leq V(x) \leq c_4|x|^{\theta_3} \log^{\theta_4}(1 + |x|).$$

*If  $\theta_1 = 1$  and  $\theta_2 > 2$  or if  $\theta_1 > 1$ , then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive, and for any  $\varepsilon > 0$ , there exists a constant  $c_5 = c_5(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$c_5 \exp \left( - \frac{(1 + \varepsilon)\theta_3}{\kappa} |x| \log(1 + |x|) \right) \leq \phi_1(x).$$

*Additionally, if (A3) also holds and*

$$J(x, y) = 0, \quad x, y \in \mathbb{R}^d \text{ with } |x - y| > \kappa,$$

*then for any  $\varepsilon > 0$ , there exists a constant  $c_6 = c_6(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$\phi_1(x) \leq c_6 \exp \left( - \frac{(1 - \varepsilon)\theta_1}{\kappa} |x| \log(1 + |x|) \right).$$

To show that Theorem 1.3 is sharp, we have the following example, which, as mentioned above, can not be studied by the method used in [15, 13, 14].

**Example 1.4.** *Suppose that assumptions (A1), (A2) and (A3) hold, and*

$$J(x, y) \asymp |x - y|^{-d-\alpha} \mathbb{1}_{\{|x-y| \leq 1\}} + e^{-|x-y|^\gamma} \mathbb{1}_{\{|x-y| > 1\}},$$

*where  $\alpha \in (0, 2)$  and  $\gamma \in (1, \infty]$ . If  $V(x) = |x|^\theta$  for some constant  $\theta > 0$ , then the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if and only if  $\theta > 1$ . When  $\theta > 1$ , we have the following explicit two-sided estimates for the ground state  $\phi_1$ .*

- (1) *If  $\gamma = \infty$ , i.e. the associated Hunt process  $(X_t)_{t \geq 0}$  is with finite range jumps, then for any  $\varepsilon \in (0, 1)$ , there exist  $c_i = c_i(\varepsilon, \theta)$  ( $i = 1, 2$ ) such that for all  $x \in \mathbb{R}^d$ ,*

$$(1.9) \quad \begin{aligned} c_1 \exp \left( - (1 + \varepsilon)\theta |x| \log(1 + |x|) \right) &\leq \phi_1(x) \\ &\leq c_2 \exp \left( - (1 - \varepsilon)\theta |x| \log(1 + |x|) \right). \end{aligned}$$

- (2) *If  $1 < \gamma < \infty$ , then there exist positive constants  $c_i := c_i(\gamma)$  ( $i = 4, 6$ ) independent of  $\theta$  such that for all  $x \in \mathbb{R}^d$ ,*

$$(1.10) \quad \begin{aligned} c_3 \exp \left( - c_4 \theta^{\frac{\gamma-1}{\gamma}} |x| \log^{\frac{\gamma-1}{\gamma}}(1 + |x|) \right) &\leq \phi_1(x) \\ &\leq c_5 \exp \left( - c_6 \theta^{\frac{\gamma-1}{\gamma}} |x| \log^{\frac{\gamma-1}{\gamma}}(1 + |x|) \right) \end{aligned}$$

*holds for some positive constants  $c_3 = c_3(\theta, \gamma)$  and  $c_5(\theta, \gamma)$ .*

We make some comments on Theorem 1.3 and Example 1.4.



**Remark 1.5.** (1) Compared with [15, 13, 14], to ensure the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  Theorem 1.3 gets rid of the following restrictive condition on potential function  $V$ :

$$(1.11) \quad \sup_{z \in B(x,1)} V(z) \leq CV(x), \quad |x| \geq 1,$$

see e.g. [14, Assumption 2.5 and Corollary 2.3(1)]. Intuitively, regularity condition (1.11) means that the rate for the oscillation of  $V$  is mild. However, according to (1.8), we know from Theorem 1.3 that  $(T_t^V)_{t \geq 0}$  still may be intrinsically ultracontractive without such regular condition on  $V$ . The reader can refer to Proposition 4.1 below for more general conditions on  $V$ . Roughly speaking, the upper bound for  $V$  in (1.8) is used to control the lower bound for the ground state  $\phi_1$ , while the lower bound for  $V$  is needed to establish the upper bound estimate for  $\phi_1$ , and also the intrinsic (local) super Poincaré inequality for Dirichlet form  $(D, \mathcal{D}(D))$ .

(2) In Lévy case, if  $V(x) = |x|^\theta$  for some  $\theta > 0$ , the conclusion of Example 1.4 says that  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if and only if  $\theta > 1$ . Such condition on  $V$  is the same as that in case of  $\gamma = 1$ , which is associated with the Feynman-Kac semigroup for relativistic  $\alpha$ -stable processes, see [15, Theorem 1.6 and the remark below] for more details. However, the case  $\gamma \in (1, \infty]$  does not fit the framework of [15, 13, 14], and it is essentially different from the case  $\gamma \in (0, 1)$ . Indeed, let  $\rho$  be the density function of the Lévy measure. According to [14, Assumption 2.1], the function  $\rho$  is required to satisfy that

- (i) There exists a constant  $C_1 > 0$  such that for every  $1 \leq |y| \leq |x|$ ,

$$\rho(y) \leq C_1 \rho(x).$$

- (ii) There exists a constant  $C_2 > 0$  such that for all  $x, y \in \mathbb{R}^d$  with  $|x - y| > 1$ ,

$$\int_{\{|z-x| \geq 1, |z-y| \geq 1\}} \rho(x-z)\rho(z-y) dz \leq C_2 \rho(x-y).$$

By [14, Example 4.1 (3)], the assumptions (i) and (ii) are only satisfied when  $\gamma \in (0, 1]$ . On the other hand, the difference between  $\gamma > 1$  and  $0 < \gamma \leq 1$  is also indicated by [7, Theorem 1.2 (1) and (2)], where explicit global heat kernel estimates of the associated process (depending on the parameter  $\gamma$ ) are presented.

(3) In Example 1.4 (1), i.e.  $\gamma = \infty$ , the symmetric Hunt process associated with density function  $J$  above is the truncated symmetric  $\alpha$ -stable-like process, e.g. see [6]. On the other hand, if the Hunt process is a Brownian motion and  $V(x) = |x|^\theta$  for some  $\theta > 0$ , then, according to [11, Theorem 6.1] (at least in one dimension case), we know that the associated Feynman-Kac semigroup is intrinsically ultracontractive if and only if  $\theta > 2$ . This, along with Example 1.4 (1), indicates the difference of the intrinsic ultracontractivity for Feynman-Kac semigroups between Lévy process (symmetric jump processes) with finite range jumps and Brownian motion.

**1.2.2. The case that  $\lim_{|x| \rightarrow \infty} V(x) \neq \infty$ .** The following theorem gives us sufficient conditions on the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  for a class of irregular potential functions  $V$  such that  $\lim_{|x| \rightarrow \infty} V(x) \neq \infty$ . Denote by  $|A|$  Lebesgue measure of a Borel set  $A \subseteq \mathbb{R}^d$ .

**Theorem 1.6.** *Suppose that (1.2), (1.3), assumptions (A1) and (A2) hold, and that there exists a unbounded subset  $A \subseteq \mathbb{R}^d$  such that the following conditions are satisfied.*

(1)  $|A| < \infty$  and

$$A \cap \{x \in \mathbb{R}^d : |x| \geq R\} \neq \emptyset, \quad \forall R > 0.$$

(2) *There exist positive constants  $c_i$  ( $i = 3, 4$ ),  $\theta_i$  ( $i = 1, 2$ ) with  $\theta_1 > 2$  and constant  $\theta_3 \in \mathbb{R}$  such that for all  $x \in \mathbb{R}^d$  with  $|x|$  large enough,*

$$V(x) = 1, \quad x \in A$$

and

$$c_3|x| \log^{\theta_1}(1 + |x|) \leq V(x) \leq c_4|x|^{\theta_2} \log^{\theta_3}(1 + |x|), \quad x \notin A.$$

(3) *There exist positive constants  $c_i$  ( $i = 5, 6$ ) and  $\eta_i$  ( $i = 1, 2$ ) such that for every  $R > 2$ ,*

$$|\{x \in \mathbb{R}^d : x \in A, |x| \geq R\}| \leq c_5 \exp(-c_6 R^{\eta_1} \log^{\eta_2} R).$$

Then, we have

- (i) *If  $\eta_1 = 1$  and  $\eta_2 > 1$ , then the associated Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.*
- (ii) *If  $\eta_1 = \eta_2 = 1$ , then there exists a constant  $c_0 > 0$  such that for any  $c_6 > c_0$ , the associated semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.*

Suppose moreover  $d > \alpha_1$ , and replace (3) by the following weaker condition

(4) *There exist positive constants  $c_7$  and  $\eta_3$  such that for every  $R > 2$ ,*

$$(1.12) \quad |\{x \in \mathbb{R}^d : x \in A, |x| \geq R\}| \leq \frac{c_7}{R^{d/\alpha_1} \log^{\eta_3} R}.$$

Then, if  $d > \alpha_1$ , (1), (2) and (4) hold with  $\eta_3 > 2d/\alpha_1$ , then the associated semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

The following example shows that one can not replace the decay rate  $d/\alpha_1$  in (1.12) by  $d/\alpha_1 - \varepsilon$  with any  $\varepsilon > 0$ .

**Example 1.7.** *Consider the truncated symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$  with some  $0 < \alpha < 2$ , i.e.*

$$J(x, y) = |x - y|^{-d-\alpha}, \quad 0 < |x - y| \leq 1$$

and

$$J(x, y) = 0, \quad |x - y| > 1.$$

For any  $\varepsilon \in (0, 1)$ , let  $A = \bigcup_{n=1}^{\infty} B(x_n, r_n)$  be such that  $x_n \in \mathbb{R}^d$  with  $|x_n| = n^{k_0}$  and  $r_n = n^{-\frac{k_0}{\alpha} + \frac{1}{d}}$  for  $n \geq 1$ , where  $k_0 > \frac{2}{\varepsilon}$ . Suppose that

$$V(x) = \begin{cases} 1, & \text{if } x \in A, \\ |x|^\theta, & \text{if } x \notin A \end{cases}$$

with some constant  $\theta > 1$ . Then  $(T_t^V)_{t \geq 0}$  is not intrinsically ultracontractive.

However, there is a constant  $c_0 > 0$  such that

$$(1.13) \quad |\{x \in \mathbb{R}^d : x \in A, |x| \geq R\}| \leq \frac{c_0}{R^{\frac{d}{\alpha} - \varepsilon}}, \quad R > 2.$$



The remainder of this paper is arranged as follows. In Section 2, we will present sufficient conditions for the intrinsic ultracontractivity of Feynman-Kac semigroup in terms of intrinsic super Poincaré inequality, see Theorem 2.1. These conditions are interesting of themselves, and they work for general framework including local Dirichlet forms and non-local Dirichlet forms. Section 3 is devoted to applying Theorem 2.1 to yield general results about the intrinsic ultracontractivity of the Feynman-Kac semigroups for non-local Dirichlet forms. We use the probabilistic method and the iterated approach to derive an explicit lower bound estimate for ground state of the semigroup  $(T_t^V)_{t \geq 0}$ , e.g. Proposition 3.4. The intrinsic local super Poincaré inequality for the Dirichlet form  $(D^V, \mathcal{D}(D^V))$  is established in Proposition 3.5. Proofs of all the statements in Section 1 are presented in Section 4, and proofs of Propositions 1.1 and 1.2 are given in Appendix.

**Notation** Throughout this paper, let  $d \geq 1$ . By  $|x|$  we denote the Euclidean norm of  $x \in \mathbb{R}^d$ , and by  $|A|$  the Lebesgue measure of a Borel set  $A$ . Denote by  $B(x, r)$  the ball with center  $x \in \mathbb{R}^d$  and radius  $r > 0$ . For any  $A, B \subset \mathbb{R}^d$ , let  $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ . We will write  $C = C(\kappa, \delta, \varepsilon, \lambda, \dots)$  to indicate the dependence of the constant  $C$  on parameters. The constants may change their values from one line to the next, even on the same line in the same formula. Let  $B_b(\mathbb{R}^d)$  be the set of bounded measurable functions on  $\mathbb{R}^d$ . For any measurable functions  $f, g$  and any  $\sigma$ -finite measure  $\mu$  on  $\mathbb{R}^d$ , we set  $\langle f, g \rangle_{L^2(\mathbb{R}^d; \mu)} := \int f(x)g(x) \mu(dx)$ , and for any  $p \in [1, \infty)$ ,  $\|f\|_{L^p(\mathbb{R}^d; \mu)} := \left( \int |f(x)|^p \mu(dx) \right)^{1/p}$ . Denote by  $\|f\|_\infty$  the  $L^\infty(\mathbb{R}^d; dx)$ -norm for any bounded function  $f$ . For any increasing function  $f$  on  $(0, \infty)$ ,  $f^{-1}(r) := \inf\{s > 0 : f(s) \geq r\}$  is its right inverse.

## 2. INTRINSIC ULTRACONTRACTIVITY FOR GENERAL DIRICHLET FORMS

The aim of this section is to present sufficient conditions for intrinsic ultracontractivity of Feynman-Kac semigroup associated with general symmetric Dirichlet forms (including local Dirichlet forms). Since we believe that the result below is interesting of itself and has wide applications, for sake of self-containedness we first introduce some necessary notations even if they are repeated by previous section.

Let  $(D, \mathcal{D}(D))$  be a regular symmetric Dirichlet form (not necessarily non-local) on  $L^2(\mathbb{R}^d, dx)$  with core  $C_c^2(\mathbb{R}^d)$ , and let  $V$  be a locally bounded non-negative measurable function on  $\mathbb{R}^d$ . Consider the following regular Dirichlet form with killing on  $L^2(\mathbb{R}^d, dx)$ :

$$D^V(f, f) = D(f, f) + \int f^2(x)V(x) dx, \quad \mathcal{D}(D^V) = \overline{C_c^2(\mathbb{R}^d)}^{D_1^V},$$

where

$$D_1^V(f, f) := D^V(f, f) + \|f\|_{L^2(\mathbb{R}^d; dx)}^2.$$

Denote by  $(T_t^V)_{t \geq 0}$  the associated (Feynman-Kac) semigroup on  $L^2(\mathbb{R}^d, dx)$ . To consider the intrinsic ultracontractivity of Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$ , we assume that

- (A) The Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  is compact on  $L^2(\mathbb{R}^d, dx)$ , and its ground state  $\phi_1$  corresponding to the first eigenvalue  $\lambda_1 > 0$  is bounded, continuous and strictly positive.
- (B) The potential function  $V$  satisfies
- (A2) For every  $r > 0$ ,

$$|\{x \in \mathbb{R}^d : V(x) \leq r\}| < \infty.$$

(A4) There exists a constant  $K > 0$  such that

$$\lim_{R \rightarrow \infty} \Phi(R) = \infty,$$

where

$$\Phi(R) = \Phi_K(R) := \inf_{|x| \geq R, V(x) > K} V(x), \quad R > 0.$$

According to assumption (A2),  $|\{x \in \mathbb{R}^d : V(x) \leq K\}| < \infty$ . Therefore, assumption (A4) means that the potential function  $V$  tends to infinity as  $|x| \rightarrow \infty$  on the complement of a set (maybe unbounded) with finite Lebesgue measure. Obviously both (A2) and (A4) hold true when

$$(2.14) \quad \lim_{|x| \rightarrow \infty} V(x) = \infty.$$

For the constant  $K$  in assumption (A4), let

$$\Theta(R) = \Theta_K(R) := |\{x \in \mathbb{R}^d : |x| \geq R, V(x) \leq K\}|, \quad R > 0.$$

On the other hand, due to the fact  $|\{x \in \mathbb{R}^d : V(x) \leq K\}| < \infty$ , it is easy to see that

$$\lim_{R \rightarrow \infty} \Theta(R) = 0.$$

In particular, if (2.14) holds, then for any constant  $K > 0$ ,  $\Theta(R) = 0$  when  $R > 0$  is large enough.

Now, we state the main result in this section.

**Theorem 2.1.** *Let  $(T_t^V)_{t \geq 0}$  be a compact Feynman-Kac semigroup on  $L^2(\mathbb{R}^d, dx)$ , and  $V$  be a locally bounded non-negative measurable function such that Assumptions (A) and (B) are satisfied. Suppose that there exists a bounded measurable function  $\varphi \in B_b(\mathbb{R}^d)$  such that the following conditions are satisfied.*

- (1) *There is a constant  $r_0 > 0$  such that for every  $r \geq r_0$ , the following local intrinsic super Poincaré inequality*

$$\begin{aligned} \int_{B(0,r)} f^2(x) dx &\leq s D^V(f, f) \\ &+ \alpha(r, s) \left( \int |f|(x) \varphi(x) dx \right)^2, \quad s > 0, f \in \mathcal{D}(D^V) \end{aligned}$$

*holds for some positive measurable function  $\alpha$ .*

- (2) *Let  $\phi_1$  be the ground state for the semigroup  $(T_t^V)_{t \geq 0}$ . It holds for some constant  $C_0 > 0$  that*

$$\varphi(x) \leq C_0 \phi_1(x), \quad x \in \mathbb{R}^d.$$

Then, the following intrinsic mixed type super Poincaré inequality

$$(2.15) \quad \int f^2(x) dx \leq sD^V(f, f) + \beta(s \wedge s_0) \left( \int |f|(x) \phi_1(x) dx \right)^2 + \gamma(s \wedge s_0)^{(p-2)/p} \|f\|_{L^p(\mathbb{R}^d; dx)}^2$$

holds for all  $s > 0$ ,  $f \in \mathcal{D}(D^V)$  and  $p \in (2, \infty]$  with the rate functions

$$(2.16) \quad \beta(s) := C_0^2 \alpha \left( \Phi^{-1} \left( \frac{2}{s} \right), \frac{s}{2} \right), \quad \gamma(s) := \Theta \left( \Phi^{-1} \left( \frac{2}{s} \right) \right)$$

and constant  $s_0 = \frac{2}{\Phi(r_0)}$ . Here, we use the convention that  $(p-2)/p = 1$  when  $p = \infty$ .

Moreover, we have

(i) If (2.14) holds, then the following super Poincaré inequality

$$(2.17) \quad \int f^2(x) dx \leq sD^V(f, f) + \beta(s \wedge r_1) \left( \int |f|(x) \phi_1(x) dx \right)^2, \quad s > 0, f \in \mathcal{D}(D^V)$$

holds for some constant  $r_1 > 0$ . Consequently, if

$$\int_t^\infty \frac{\beta^{-1}(s)}{s} ds < \infty, \quad t > \inf \beta,$$

then the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

(ii) If for some  $p > 2$  there is a constant  $c_0 > 0$  such that the following Sobolev inequality holds true

$$(2.18) \quad \|f\|_{L^p(\mathbb{R}^d; dx)}^2 \leq c_0 \left[ D^V(f, f) + \|f\|_{L^2(\mathbb{R}^d; dx)}^2 \right], \quad f \in C_c^\infty(\mathbb{R}^d),$$

then the super Poincaré inequality (2.17) holds with the rate function  $\beta$  and the constant  $r_1$  replaced by

$$(2.19) \quad \hat{\beta}(s) := 2C_0^2 \alpha \left( \Psi^{-1} \left( \frac{s}{4} \right), \frac{s}{4} \right)$$

and some constant  $r_2 > 0$  respectively, where

$$\Psi(R) := \frac{1}{\Phi(R)} + c_0 \Theta(R)^{\frac{p-2}{p}}, \quad R > 1.$$

Consequently, if

$$\int_t^\infty \frac{\hat{\beta}^{-1}(s)}{s} ds < \infty, \quad t > \inf \hat{\beta},$$

then the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

(iii) If there exists a constant  $\delta > 1$  such that

$$(2.20) \quad \sum_{n=1}^{\infty} \gamma(s_n) \delta^n < \infty,$$

where  $s_n := \beta^{-1}(\frac{c_1 \delta^n}{2})$  with  $c_1 := \|\phi_1\|_\infty^2$ , then the following super Poincaré inequality holds

$$(2.21) \quad \int f^2(x) dx \leq_s D^V(f, f) + \tilde{\beta}(s \wedge r_2) \left( \int |f|(x) \phi_1(x) dx \right)^2, \quad s > 0, f \in \mathcal{D}(D^V),$$

where  $r_2$  is a positive constant,

$$\tilde{\beta}(s) := 2\beta \left( \gamma^{-1} \left( \frac{1}{4\delta^{n_0(s)+1}} \right) \right)$$

and

$$(2.22) \quad n_0(s) := \inf \left\{ N \geq \left( \log_\delta \left( \frac{2\beta(s_0)}{c_1} \right) \right) \vee (-\log_\delta(4\delta\gamma(s_0))) : \frac{4\delta(\sqrt{\delta}+1)s_N}{\sqrt{\delta}-1} + 2\gamma^{-1} \left( \frac{1}{4\delta^{N+1}} \right) \leq s \right\}.$$

Consequently, if

$$\int_t^\infty \frac{\tilde{\beta}^{-1}(s)}{s} ds < \infty, \quad t > \inf \tilde{\beta},$$

then the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

*Proof.* Throughout the proof, we denote by  $\mu$  the Lebesgue measure on  $\mathbb{R}^d$ . For the constant  $K$  in assumption **(A4)**, let  $A_1 := \{x \in \mathbb{R}^d : V(x) > K\}$  and  $A_2 := \mathbb{R}^d \setminus A_1$ . For any  $f \in \mathcal{D}(D^V)$ ,  $R \geq r_0$  and  $p \in (2, \infty]$ , it holds that

$$(2.23) \quad \begin{aligned} \int_{B(0,R)^c} f^2(x) \mu(dx) &= \int_{B(0,R)^c \cap A_1} f^2(x) \mu(dx) + \int_{B(0,R)^c \cap A_2} f^2(x) \mu(dx) \\ &\leq \frac{1}{\Phi(R)} \int_{B(0,R)^c \cap A_1} f^2(x) V(x) \mu(dx) \\ &\quad + \mu(B(0,R)^c \cap A_2)^{(p-2)/p} \|f\|_{L^p(\mathbb{R}^d; dx)}^2 \\ &\leq \frac{1}{\Phi(R)} D^V(f, f) + \Theta(R)^{(p-2)/p} \|f\|_{L^p(\mathbb{R}^d; dx)}^2, \end{aligned}$$

where in the first inequality we have used the Hölder inequality when  $p \in (2, \infty)$ . This, along with conditions (1) and (2), gives us that for any  $R, \tilde{s} > 0$  and  $f \in \mathcal{D}(D^V)$ ,

$$\mu(f^2) \leq \left( \frac{1}{\Phi(R)} + \tilde{s} \right) D^V(f, f) + C_0^2 \alpha(R, \tilde{s}) \mu(\phi_1 |f|)^2 + \Theta(R)^{(p-2)/p} \|f\|_{L^p(\mathbb{R}^d; dx)}^2.$$

For any  $0 < s \leq s_0 := \frac{2}{\Phi(r_0)}$ , taking  $R = \Phi^{-1}(\frac{2}{s})$  and  $\tilde{s} = \frac{s}{2}$  in the inequality above, we can get the required mixed type super Poincaré inequality (2.15) for all  $s \in (0, s_0]$ . Hence, the proof of the first assertion is completed by choosing  $\beta(s) = \beta(s_0)$  and  $\gamma(s) = \gamma(s_0)$  for all  $s \geq s_0$ .

(i) We take  $p = \infty$  in (2.23). Suppose that (2.14) holds. Then  $\Theta(R) = 0$  for  $R > 0$  large enough. This immediately yields the true super Poincaré inequality (2.17) with some constant  $r_1 > 0$ .

Let  $(L^V, \mathcal{D}(L^V))$  be the generator associated with  $(T_t^V)_{t \geq 0}$ , and  $(\tilde{T}_t^V)_{t \geq 0}$  be the strongly continuous semigroup defined by (1.5). Due to the fact that  $L_V \phi_1 = -\lambda_1 \phi_1$ , the (regular) Dirichlet form  $(D_{\phi_1}, \mathcal{D}(D_{\phi_1}))$  associated with  $(\tilde{T}_t^V)_{t \geq 0}$  enjoys the properties that,  $C_c^2(\mathbb{R}^d)$  is a core for  $(D_{\phi_1}, \mathcal{D}(D_{\phi_1}))$ , and for any  $f \in C_c^2(\mathbb{R}^d)$ ,

$$(2.24) \quad D_{\phi_1}(f, f) = D^V(f\phi_1, f\phi_1) - \lambda_1 \int_{\mathbb{R}^d} f^2(x) \phi_1^2(x) dx.$$

Let  $\mu_{\phi_1}(dx) = \phi_1^2(x) dx$ . Combining (2.24) with (2.17) gives us the following intrinsic super Poincaré inequality

$$\begin{aligned} \mu_{\phi_1}(f^2) &\leq s D^V(f\phi_1, f\phi_1) + \beta(s \wedge r_1) \left( \int |f|(x) \phi_1^2(x) dx \right)^2 \\ &\leq s \left( D_{\phi_1}(f, f) + \lambda_1 \mu_{\phi_1}(f^2) \right) + \beta(s \wedge r_1) \mu_{\phi_1}^2(|f|), \quad s > 0, \end{aligned}$$

where the rate function  $\beta(s)$  is given by (2.16). In particular, for any  $s \in (0, 1/(2\lambda_1))$ ,

$$\mu_{\phi_1}(f^2) \leq 2s D_{\phi_1}(f, f) + 2\beta(s \wedge r_1), \quad f \in C_c^2(\mathbb{R}^d),$$

which implies that

$$\mu_{\phi_1}(f^2) \leq s D_{\phi_1}(f, f) + 2\beta\left(\frac{s}{2} \wedge r_1 \wedge \frac{1}{\lambda_1}\right), \quad f \in C_c^2(\mathbb{R}^d), s > 0.$$

Therefore, the desired assertion in (i) for the ultracontractivity of the (Markovian) semigroup  $(\tilde{T}_t^V)_{t \geq 0}$  (or, equivalently, the intrinsic ultracontractivity of the semigroup  $(T_t^V)_{t \geq 0}$ ) follows from [19, Theorem 3.3.13] or [16, Theorem 3.1].

(ii) Suppose (2.18) holds true. According to (2.23) and (2.18), for any  $f \in C_c^\infty(\mathbb{R}^d)$  and  $R \geq r_0$ ,

$$\int_{B(0, R)^c} f^2(x) \mu(dx) \leq \left( \frac{1}{\Phi(R)} + c_0 \Theta(R)^{(p-2)/p} \right) D^V(f, f) + c_0 \Theta(R)^{(p-2)/p} \mu(f^2),$$

Combining this with conditions (1) and (2) in Theorem 2.1, we have that for any  $R \geq R_1 \vee r_0$  with  $c_0 \Theta(R_1)^{(p-2)/p} \leq 1/2$ , each  $\tilde{s} > 0$  and  $f \in C_c^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} \mu(f^2) &\leq 2 \left( \frac{1}{\Phi(R)} + c_0 \Theta(R)^{(p-2)/p} + \tilde{s} \right) D^V(f, f) + 2C_0^2 \alpha(R, \tilde{s}) \mu(\phi_1 |f|)^2 \\ &= 2(\Psi(R) + \tilde{s}) D^V(f, f) + 2C_0^2 \alpha(R, \tilde{s}) \mu(\phi_1 |f|)^2. \end{aligned}$$

Taking  $R = \Psi^{-1}(\frac{s}{4})$  and  $\tilde{s} = \frac{s}{4}$  in the inequality above for  $s > 0$  small enough, we can get the super Poincaré inequality (2.17) with the desired rate function  $\hat{\beta}$  given by (2.19), also thanks to the fact that  $C_c^\infty(\mathbb{R}^d)$  is dense in  $\mathcal{D}(D^V)$ . Having this at hand, we can arrive at the final assertion for (ii) by the same argument as that in the proof of part (i).

(iii) Now we take  $p = \infty$  in (2.23), and assume that (2.20) holds. Note that, for every  $f \in \mathcal{D}(D^V)$ ,  $|f| \in \mathcal{D}(D^V)$  and  $D^V(|f|, |f|) \leq D^V(f, f)$ , and so it suffices to prove that (2.21) holds for any  $f \in \mathcal{D}(D^V)$  with  $f \geq 0$ .

Given any  $f \in \mathcal{D}(D^V)$  with  $f \geq 0$  and  $\mu(f^2) = 1$ , for any  $\delta > 1$ , we have

$$\begin{aligned}
 \mu(f^2) &= \int_0^\infty \mu(f^2 > t) dt \\
 &= \int_0^{\delta^{n_0+1}} \mu(f^2 > t) dt + \sum_{n=n_0+1}^\infty \int_{\delta^n}^{\delta^{n+1}} \mu(f^2 > t) dt \\
 (2.25) \quad &\leq \mu((f \wedge \delta^{\frac{n_0+1}{2}})^2) + \sum_{n=n_0+1}^\infty (\delta^{n+1} - \delta^n) \mu(f^2 > \delta^n) \\
 &=: J_{n_0} + \sum_{n=n_0+1}^\infty I_n,
 \end{aligned}$$

where  $n_0$  is an integer to be determined later.

Next, we define

$$f_n := (f - \delta^{\frac{n}{2}})^+ \wedge (\delta^{\frac{n+1}{2}} - \delta^{\frac{n}{2}}), \quad n \geq 0.$$

Noticing that for any  $n \geq 0$ ,

$$f_n \geq (\delta^{\frac{n+1}{2}} - \delta^{\frac{n}{2}}) \mathbf{1}_{\{f^2 > \delta^{n+1}\}},$$

we get

$$\begin{aligned}
 I_n &= (\delta^{n+1} - \delta^n) \mu(f^2 > \delta^n) \\
 (2.26) \quad &\leq \frac{(\delta^{n+1} - \delta^n) \mu(f_{n-1}^2)}{(\delta^{\frac{n}{2}} - \delta^{\frac{n-1}{2}})^2} \\
 &= \frac{\delta(\sqrt{\delta} + 1)}{\sqrt{\delta} - 1} \mu(f_{n-1}^2), \quad n \geq 0.
 \end{aligned}$$

According to (2.15) and the fact that  $f_n \in \mathcal{D}(D^V)$ , for all  $n \geq 0$  and  $0 < s \leq s_0$ ,

$$(2.27) \quad \mu(f_n^2) \leq s D^V(f_n, f_n) + \beta(s) \mu(\phi_1 f_n)^2 + \gamma(s) \delta^n (\sqrt{\delta} - 1)^2.$$

Due to the Cauchy-Schwarz inequality, the Chebyshev inequality and the fact that  $\mu(f^2) = 1$ ,

$$\mu(\phi_1 f_n)^2 = \mu(\phi_1 f_n \mathbf{1}_{\{f^2 > \delta^n\}})^2 \leq \mu(\phi_1^2 \mathbf{1}_{\{f^2 > \delta^n\}}) \mu(f_n^2) \leq c_1 \delta^{-n} \mu(f_n^2).$$

Then, taking  $s = s_n := \beta^{-1}(\frac{c_1 \delta^n}{2})$  with  $n \geq \log_\delta \left( \frac{2\beta(s_0)}{c_1} \right)$  in (2.27), we obtain

$$\mu(f_n^2) \leq s_n D^V(f_n, f_n) + \frac{1}{2} \mu(f_n^2) + (\sqrt{\delta} - 1)^2 \gamma(s_n) \delta^n,$$

which implies that

$$\mu(f_n^2) \leq 2s_n D^V(f_n, f_n) + 2(\sqrt{\delta} - 1)^2 \gamma(s_n) \delta^n.$$

Since (2.20) holds true, there exists an integer

$$n_0 \geq \left( \log_\delta \left( \frac{2\beta(s_0)}{c_1} \right) \right) \vee (-\log_\delta(4\delta\gamma(s_0)))$$

such that

$$\delta(\delta - 1) \sum_{i=n_0}^\infty \gamma(s_i) \delta^i \leq \frac{1}{8}.$$



Furthermore, it is easy to see that

$$\sum_{n=0}^{\infty} |f_n(x) - f_n(y)| \leq |f(x) - f(y)|, \quad \sum_{n=0}^{\infty} |f_n(x)| \leq |f(y)|,$$

so, according to [19, Lemma 3.3.2],

$$\sum_{n=0}^{\infty} D^V(f_n, f_n) \leq D^V(f, f).$$

Combining all the estimates above with (2.26), and noting that  $s_n$  is non-increasing with respect to  $n$ , we arrive at

$$(2.28) \quad \sum_{n=n_0+1}^{\infty} I_n \leq \frac{2\delta(\sqrt{\delta} + 1)s_{n_0}}{\sqrt{\delta} - 1} D^V(f, f) + \frac{1}{4}.$$

On the other hand, applying  $f \wedge \delta^{\frac{n_0+1}{2}}$  into (2.17), we have

$$\begin{aligned} J_{n_0} &= \mu((f \wedge \delta^{\frac{n_0+1}{2}})^2) \\ &\leq sD^V(f \wedge \delta^{\frac{n_0+1}{2}}, f \wedge \delta^{\frac{n_0+1}{2}}) + \beta(s)\mu(\phi_1 f)^2 + \gamma(s)\delta^{n_0+1} \\ &\leq sD^V(f, f) + \beta(s)\mu(\phi_1 f)^2 + \gamma(s)\delta^{n_0+1}, \quad 0 < s \leq s_0, \end{aligned}$$

where the second inequality also follows from [19, Lemma 3.3.2]. Hence, noticing that  $n_0 \geq -\log_{\delta}(4\delta\gamma(s_0))$  and taking  $s = \gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)$  in the inequality above, we get that

$$(2.29) \quad J_{n_0} \leq \gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right) D^V(f, f) + \beta\left(\gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)\right) \mu(\phi_1 f)^2 + \frac{1}{4}.$$

According to (2.25), (2.28) and (2.29), we obtain

$$\begin{aligned} \mu(f^2) &\leq \left(\frac{2\delta(\sqrt{\delta} + 1)s_{n_0}}{\sqrt{\delta} - 1} + \gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)\right) D^V(f, f) \\ &\quad + \beta\left(\gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)\right) \mu(\phi_1 f)^2 + \frac{1}{2}. \end{aligned}$$

Since  $\mu(f^2) = 1$ , this implies that

$$\begin{aligned} \mu(f^2) &\leq \left(\frac{4\delta(\sqrt{\delta} + 1)s_{n_0}}{\sqrt{\delta} - 1} + 2\gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)\right) D^V(f, f) \\ &\quad + 2\beta\left(\gamma^{-1}\left(\frac{1}{4\delta^{n_0+1}}\right)\right) \mu(\phi_1 f)^2 \end{aligned}$$

Hence, for  $s > 0$  small enough, we arrive at the desired super Poincaré inequality by taking  $n_0$  to be  $n_0(s)$  defined by (2.22). The intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  is easily verified by following the argument of (i).  $\square$

**Remark 2.2.** To derive the intrinsic ultracontractivity of Feynman-Kac semigroups (or Dirichlet semigroups), Rosen's lemma in the context of super log-Sobolev inequality was applied in [9, 11]. Instead of such approach, in Theorem 2.1 we use super Poincaré inequality. The main advantage of our method is due to that, a mixed type super Poincaré inequality can be applied to study the situation that Sobolev

inequality (2.18) fails, e.g. a symmetric  $\alpha$ -stable process on  $\mathbb{R}$  with  $\alpha \geq 1$ , for which the method of [9] does not work.

In Theorem 2.1, we essentially use the lower bound estimate for the ground state. On the contrary, at the end of this section we present the following sufficient conditions for an upper bound estimate of the ground state.

**Proposition 2.3.** *Suppose that the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive. Let  $(L^V, \mathcal{D}(L^V))$  be the generator associated with  $(T_t^V)_{t \geq 0}$ . If there exist a positive function  $\psi \in C_b^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; dx)$  and a constant  $\lambda > 0$  such that  $\psi \in \mathcal{D}(L^V)$ ,*

$$(2.30) \quad L^V \psi(x) \leq \lambda \psi(x), \quad x \in \mathbb{R}^d,$$

then there is a constant  $c_1 > 0$  such that

$$\phi_1(x) \leq c_1 \psi(x), \quad x \in \mathbb{R}^d.$$

*Proof.* Under (2.30), we know that

$$T_t^V \psi(x) \leq e^{\lambda t} \psi(x), \quad x \in \mathbb{R}^d, t > 0.$$

According to [11, Theorem 3.2], the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  implies that for every  $t > 0$ , there is a constant  $c_t > 0$  such that

$$p^V(t, x, y) \geq c_t \phi_1(x) \phi_1(y), \quad x, y \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned} \psi(x) &\geq e^{-\lambda} T_1^V \psi(x) = e^{-\lambda} \int p^V(1, x, y) \psi(y) dy \\ &\geq c_1 e^{-\lambda} \int \psi(y) \phi_1(y) dy \phi_1(x) =: C_0 \phi_1(x), \end{aligned}$$

which yields the required assertion.  $\square$

### 3. INTRINSIC ULTRA CONTRACTIVITY FOR NON-LOCAL DIRICHLET FORMS

In this section, we come back to the framework introduced in Subsection 1.1 which is recalled below. Let  $(D, \mathcal{D}(D))$  be the non-local Dirichlet form given in (1.1) such that jump kernel satisfies (1.2) and (1.3), i.e., there exist  $\alpha_1, \alpha_2 \in (0, 2)$  with  $\alpha_1 \leq \alpha_2$  and positive  $c_1, c_2, \kappa$  such that

$$c_1 |x - y|^{-d-\alpha_1} \leq J(x, y) \leq c_2 |x - y|^{-d-\alpha_2}, \quad 0 < |x - y| \leq \kappa$$

and

$$\sup_{x \in \mathbb{R}^d} \int_{\{|y-x| > \kappa\}} J(x, y) dy < \infty.$$

Assume that the corresponding symmetric Hunt process  $((X_t)_{t \geq 0}, \mathbb{P}^x)$  is well defined for all  $x \in \mathbb{R}^d$ , and that the process  $(X_t)_{t \geq 0}$  possesses a positive, bounded and continuous density function  $p(t, x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$  for all  $t > 0$ . That is, assumption **(A1)** holds. Note that, when  $J(x, y) = 0$  for any  $x, y \in \mathbb{R}^d$  with  $|x - y| > \kappa$ , the process  $(X_t)_{t \geq 0}$  has finite range jumps.

Let  $(T_t)_{t \geq 0}$  be a symmetric semigroup associated with  $(D, \mathcal{D}(D))$ , i.e.

$$T_t(f)(x) = \mathbb{E}^x(f(X_t)), \quad x \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d; dx).$$

Let  $V$  be a non-negative measurable and locally bounded potential function on  $\mathbb{R}^d$  such that Assumption **(A2)** is satisfied. Define the Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  associated with the Hunt process  $(X_t)_{t \geq 0}$  as follows:

$$T_t^V(f)(x) = \mathbb{E}^x \left( \exp \left( - \int_0^t V(X_s) ds \right) f(X_t) \right), \quad x \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d; dx).$$

Then, the non-local regular Dirichlet form associated with  $(T_t^V)_{t \geq 0}$  is given by

$$D^V(f, f) = \frac{1}{2} \iint (f(x) - f(y))^2 J(x, y) dx dy + \int f^2(x) V(x) dx,$$

$$\mathcal{D}(D^V) = \overline{C_c^1(\mathbb{R}^d)}^{D_1^V},$$

where  $D_1^V(f, f) := D^V(f, f) + \|f\|_{L^2(\mathbb{R}^d; dx)}^2$ . According to Propositions 1.1 and 1.2, we know that the Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  is compact on  $L^2(\mathbb{R}^d, dx)$ , and it has a bounded, continuous and strictly positive ground state  $\phi_1$  corresponding to the first eigenvalue  $\lambda_1 > 0$ .

In the following, we will apply Theorem 2.1 to establish the intrinsic ultracontractivity of Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$ . For this, we need obtain some nice lower bound estimate for the ground state  $\phi_1$ , and derive local super Poincaré inequality or Sobolev inequality for the Dirichlet form  $(D^V, \mathcal{D}(D^V))$ . These will be considered in the following three subsections respectively, and then general results about the intrinsic ultracontractivity of Feynman-Kac semigroup for non-local Dirichlet forms are presented in Subsection 3.4.

**3.1. Lower bound estimate for the ground state.** For any Borel set  $D \subseteq \mathbb{R}^d$ , let  $\tau_D := \inf\{t > 0 : X_t \notin D\}$  be the first exit time from  $D$  of the process  $(X_t)_{t \geq 0}$ . Denote by  $B(x, r)$  the ball with center at  $x \in \mathbb{R}^d$  and radius  $r > 0$ .

**Lemma 3.1.** *There exist positive constants  $c_0 := c_0(\kappa)$  and  $r_0 := r_0(\kappa)$  such that for every  $r \in (0, r_0]$  and  $x \in \mathbb{R}^d$ , we have*

$$(3.31) \quad \mathbb{P}^x \left( \tau_{B(x, r)} > c_0 r^{\alpha_2 + \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}} \right) \geq \frac{1}{2}.$$

*Proof.* For any  $0 < s < \kappa$ , set

$$L_1(s) := \sup_{x \in \mathbb{R}^d} \int_{\{|y-x| > s\}} J(x, y) dy,$$

$$L_2(s) := \sup_{x \in \mathbb{R}^d} \int_{\{|y-x| \leq s\}} |x - y|^2 J(x, y) dy,$$

$$L(s) := L_1(s) + s^d (s^{-2} L_2(s))^{\frac{(d+\alpha_1)}{\alpha_1}}.$$

According to [2, Theorem 2.1], there exists a constant  $r_0 := r_0(\kappa) > 0$  such that for every  $0 < r < r_0$ ,  $t > 0$  and  $x \in \mathbb{R}^d$ ,

$$(3.32) \quad \mathbb{P}^x (\tau_{B(x, r)} < t) \leq C_1 t L(r),$$

where  $C_1$  is a positive constant independent of  $t$  and  $r$ .

Without lose of generality, we may and do assume that  $0 < r_0 < 1$ . Then, by (1.2) and (1.3), for every  $r \in (0, r_0)$

$$L(r) \leq C_2 \left( r^{-\alpha_2 - \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}} + L_1(\kappa) \right) \leq C_3 r^{-\alpha_2 - \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}}.$$

Let  $c_0 := c_0(\kappa)$  be a positive constant such that  $c_0 C_1 C_3 \leq 1/2$ . Then the required assertion (3.31) follows from (3.32) by taking  $t = c_0 r^{\alpha_2 + \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}}$ .  $\square$

**Lemma 3.2.** *Let  $r_0, c_0$  be the constants given in Lemma 3.1, and set  $\varepsilon_0 = r_0/\kappa$ . Then, for any  $\varepsilon \in (0, \varepsilon_0)$ , any two disjoint sets  $B \supseteq B(x, \varepsilon\kappa)$  and  $D = B(y, \varepsilon\kappa)$  for some  $x, y \in \mathbb{R}^d$  satisfying that  $\text{dist}(B, D) > \varepsilon\kappa$  and  $|z_1 - z_2| \leq \kappa$  for every  $z_1 \in B$  and  $z_2 \in D$ , and every  $0 < t_1 < t_2 < T(\kappa, \varepsilon) := c_0(\varepsilon\kappa)^{\alpha_2 + \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}}$ , it holds that*

$$\mathbb{P}^x \left( X_{\tau_B} \in D, t_1 \leq \tau_B < t_2 \right) \geq c_1 \varepsilon^d \kappa^{-\alpha_1} (t_2 - t_1),$$

where  $c_1$  is a positive constant independent of  $\kappa, \varepsilon, x$  and  $y$ .

*Proof.* Denote by  $p_B(t, x, y)$  the density of the process  $(X_t)_{t \geq 0}$  killed on exiting the set  $B$ , i.e.

$$p_B(t, x, y) = p(t, x, y) - \mathbb{E}^x(\tau_B \leq t; p(t - \tau_B, X(\tau_B), y)).$$

According to the framework of the Lévy system for the Dirichlet form  $(D, \mathcal{D}(D))$  (see e.g. [5, Lemma 4.8]), we have for disjoint open sets  $B$  and  $D$  that

$$\begin{aligned} \mathbb{P}^x \left( X_{\tau_B} \in D \right) &= \mathbb{E}^x \left( \int_0^{\tau_B} \int_D J(X_s, z) dz ds \right) \\ &= \int_B \int_0^\infty p_B(s, x, y) ds \int_D J(y, z) dz dy, \quad x \in B. \end{aligned}$$

Then, following the proof of [15, Proposition 2.5], we get that

$$\mathbb{P}^x \left( X_{\tau_B} \in D, t_1 \leq \tau_B < t_2 \right) = \int_B \int_{t_1}^{t_2} p_B(s, x, y) ds \int_D J(y, z) dz dy.$$

Therefore, it holds for every  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\begin{aligned} \mathbb{P}^x \left( X_{\tau_B} \in D, t_1 \leq \tau_B < t_2 \right) &= \int_B \int_{t_1}^{t_2} p_B(s, x, y) ds \int_D J(y, z) dz dy \\ &\geq \int_B \int_{t_1}^{t_2} p_B(s, x, y) ds \int_D \frac{c_1}{|z - y|^{d+\alpha_1}} dz dy \\ &\geq C \kappa^{-d-\alpha_1} |D| \int_B \int_{t_1}^{t_2} p_B(s, x, y) ds dy \\ &= C \varepsilon^d \kappa^{-\alpha_1} \int_{t_1}^{t_2} \mathbb{P}^x(\tau_B > s) ds \\ &\geq C \varepsilon^d \kappa^{-\alpha_1} (t_2 - t_1) \mathbb{P}^x(\tau_B > T(\kappa, \varepsilon)) \\ &\geq C \varepsilon^d \kappa^{-\alpha_1} (t_2 - t_1) \mathbb{P}^x(\tau_{B(x, \varepsilon\kappa)} > T(\kappa, \varepsilon)) \\ &\geq C \varepsilon^d \kappa^{-\alpha_1} (t_2 - t_1), \end{aligned}$$

where in the first inequality we have used (1.2), the second inequality is due to  $|y - z| \leq \kappa$  for every  $y \in B$  and  $z \in D$ , and the last inequality follows from (3.31) with  $r = \varepsilon\kappa$ .  $\square$

**Lemma 3.3.** *Let  $\varepsilon_0, c_0$  be the two constants given in Lemma 3.2. For any  $\varepsilon \in (0, \min(1/11, \varepsilon_0))$ , let  $D = B(0, 2\varepsilon\kappa)$ , and  $t_0 = T(\kappa, \varepsilon) := c_0(\varepsilon\kappa)^{\alpha_2 + \frac{(\alpha_2 - \alpha_1)d}{\alpha_1}}$ . Then, there is a constant  $c_2(\kappa, \varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$  with  $|x| > \frac{\kappa(1-5\varepsilon)(1-4\varepsilon)}{\varepsilon}$ ,*

$$(3.33) \quad T_{t_0}^V(\mathbb{1}_D)(x) \geq \exp\left(-\frac{1}{(1-6\varepsilon)\kappa}|x| \log\left(1 + |x| + \sup_{|z| \leq |x| + 2\varepsilon\kappa} V(z)\right) - c_2(\kappa, \varepsilon)\right).$$

*Proof.* For any  $x \in \mathbb{R}^d$  with  $|x| > \frac{\kappa(1-5\varepsilon)(1-4\varepsilon)}{\varepsilon}$ , let

$$n = \left\lfloor \frac{1}{(1-4\varepsilon)\kappa}|x| \right\rfloor + 1$$

and  $x_i = ix/n$  for any  $0 \leq i \leq n$ , where  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ . In particular,  $x_0 = 0$ ,  $x_n = x$  and

$$\frac{1}{(1-4\varepsilon)\kappa}|x| \leq n < \frac{1}{(1-5\varepsilon)\kappa}|x|.$$

Next, for all  $0 \leq i \leq n$ , set  $D_i := B(x_i, 2\varepsilon\kappa)$ ,  $\tilde{D}_i := B(x_i, \varepsilon\kappa)$ . We can check that for all  $0 \leq i \leq n-1$ ,  $\text{dist}(D_i, D_{i+1}) > (1-5\varepsilon)\kappa - 4\varepsilon\kappa \geq 2\varepsilon\kappa$ , and  $|z_i - z_{i+1}| \leq (1-4\varepsilon)\kappa + 4\varepsilon\kappa = \kappa$  for every  $z_i \in D_i$  and  $z_{i+1} \in D_{i+1}$ .

In the following, we define for all  $n \geq 1$ ,

$$\begin{aligned} \tilde{\tau}_{D_i} &:= \inf\{t \geq \tilde{\tau}_{D_{i+1}} : X_t \notin D_i\}, \quad 1 \leq i \leq n-1; \\ \tilde{\tau}_{D_n} &:= \tau_{D_n}. \end{aligned}$$

By the convention, we also set  $\tilde{\tau}_{D_{n+1}} = 0$ . Then,

$$\begin{aligned} (3.34) \quad & T_{t_0}^V(\mathbb{1}_D)(x) \\ &= \mathbb{E}^x\left(\mathbb{1}_D(X_{t_0}) \exp\left(-\int_0^{t_0} V(X_s) ds\right)\right) \\ &\geq \mathbb{E}^x\left(0 < \tilde{\tau}_{D_i} - \tilde{\tau}_{D_{i+1}} < \frac{t_0}{n}, X_{\tilde{\tau}_{D_i}} \in \tilde{D}_{i-1} \text{ for each } 1 \leq i \leq n, \forall_{s \in [\tilde{\tau}_{D_1}, t_0]} X_s \in D; \right. \\ &\quad \left. \exp\left(-\sum_{i=1}^n \int_{\tilde{\tau}_{D_{i+1}}}^{\tilde{\tau}_{D_i}} V(X_s) ds - \int_{\tilde{\tau}_{D_1}}^{t_0} V(X_s) ds\right)\right) \\ &= \mathbb{E}^x\left(0 < \tau_{D_n} < \frac{t_0}{n}, X_{\tau_{D_n}} \in \tilde{D}_{n-1}; \exp\left(-\int_0^{\tau_{D_n}} V(X_s) ds\right) \right. \\ &\quad \cdot \mathbb{E}^{X_{\tau_{D_n}}}\left(0 < \tau_{D_{n-1}} < \frac{t_0}{n}, X_{\tau_{D_{n-1}}} \in \tilde{D}_{n-2}; \exp\left(-\int_0^{\tau_{D_{n-1}}} V(X_s) ds\right) \right. \\ &\quad \cdot \mathbb{E}^{X_{\tau_{D_{n-1}}}}\left(\dots \mathbb{E}^{X_{\tau_{D_2}}}\left(0 < \tau_{D_1} < \frac{t_0}{n}, X_{\tau_{D_1}} \in \tilde{D}_0; \exp\left(-\int_0^{\tau_{D_1}} V(X_s) ds\right) \right. \right. \\ &\quad \left. \left. \cdot \mathbb{E}^{X_{\tau_{D_1}}}\left(\forall_{s \in [0, t_0 - \tau_{D_1}]} X_s \in D; \exp\left(-\int_0^{t_0 - \tau_{D_1}} V(X_s) ds\right)\right)\right)\right)\dots\right), \end{aligned}$$

where in the last equality we have used the strong Markov property.

On the one hand, according to Lemma 3.2, for any  $2 \leq i \leq n+1$ , if  $X_{\tilde{\tau}_{D_i}} \in \tilde{D}_{i-1}$ , then for every  $i > 1$ ,

$$\mathbb{E}^{X_{\tilde{\tau}_{D_i}}}\left(0 < \tau_{D_{i-1}} < \frac{t_0}{n}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2}; \exp\left(-\int_0^{\tau_{D_{i-1}}} V(X_s) ds\right)\right)$$

$$\begin{aligned}
&\geq \sum_{j=1}^{\infty} \mathbb{E}^{X_{\tau_{D_i}}} \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2}; \exp \left( - \int_0^{\tau_{D_{i-1}}} V(X_s) ds \right) \right) \\
&\geq \sum_{j=1}^{\infty} \exp \left( - \frac{t_0}{jn} \sup_{x \in D_{i-1}} V(x) \right) \\
&\quad \times \inf_{y \in \tilde{D}_{i-1}} \mathbb{E}^y \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2} \right) \\
&\geq \frac{C\varepsilon^d \kappa^{-\alpha_1} t_0}{n} \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \exp \left( - \frac{t_0}{jn} \sup_{x \in D_{i-1}} V(x) \right) \\
&\geq \frac{C\varepsilon^d \kappa^{-\alpha_1} t_0}{n + t_0 \sup_{x \in D_{i-1}} V(x)},
\end{aligned}$$

where in the third inequality we have used Lemma 3.2 with  $B = D_{i-1}$  and  $D = \tilde{D}_{i-2}$ , and the last inequality follows from [15, Lemma 5.2], i.e.

$$(3.35) \quad \sum_{j=1}^{\infty} \frac{e^{-r/j}}{j(j+1)} \geq \frac{e^{-1}}{r+1}, \quad r \geq 0.$$

On the other hand, due to Lemma 3.1, if  $X_{\tilde{\tau}_{D_1}} \in \tilde{D}_0$ , then

$$\begin{aligned}
&\mathbb{E}^{X_{\tilde{\tau}_{D_1}}} \left( \forall_{s \in [0, t_0 - \tilde{\tau}_{D_1}]} X_s \in D; \exp \left( - \int_0^{t_0 - \tilde{\tau}_{D_1}} V(X_s) ds \right) \right) \\
(3.36) \quad &\geq \exp \left( - t_0 \sup_{z \in D} V(z) \right) \inf_{y \in \tilde{D}_0} \mathbb{E}^y \left( \tau_D > t_0 \right) \\
&\geq \exp \left( - t_0 \sup_{z \in D} V(z) \right) \inf_{y \in \tilde{D}_0} \mathbb{E}^y \left( \tau_{B(y, \varepsilon \kappa)} > T(\kappa, \varepsilon) \right) \\
&\geq C(\kappa, \varepsilon).
\end{aligned}$$

Combining all the estimates above with the fact that  $n \leq \frac{1}{(1-5\varepsilon)\kappa} |x|$ , we obtain that

$$\begin{aligned}
T_{t_0}^V(\mathbf{1}_D)(x) &\geq C(\kappa, \varepsilon) \prod_{i=1}^n \left( \frac{C\varepsilon^d \kappa^{-\alpha_1} t_0}{n + t_0 \sup_{z \in D_{i-1}} V(z)} \right) \\
&\geq C(\kappa, \varepsilon) \left( \frac{C\varepsilon^d \kappa^{-\alpha_1} t_0}{n + t_0 \sup_{|z| \leq |x| + 2\varepsilon \kappa} V(z)} \right)^n \\
&\geq \exp \left( - \frac{1}{(1-6\varepsilon)\kappa} |x| \log(1 + |x| + \sup_{|z| \leq |x| + 2\varepsilon \kappa} V(z)) - C(\kappa, \varepsilon) \right),
\end{aligned}$$

which completes the proof.  $\square$

According to Lemma 3.3, we can obtain the following lower bound estimate for the ground state.

**Proposition 3.4.** *For any  $\varepsilon \in (0, \min(1/11, \varepsilon_0))$  and  $x \in \mathbb{R}^d$ , it holds*

$$(3.37) \quad \phi_1(x) \geq \exp \left( - \frac{1}{\kappa(1-6\varepsilon)} |x| \log(1 + |x| + \sup_{|z| \leq |x| + 2\varepsilon \kappa} V(z)) - c_3(\kappa, \varepsilon) \right)$$

for some positive constant  $c_3(\kappa, \varepsilon)$  independent of  $x$ .



*Proof.* Since  $\phi_1$  is continuous and strictly positive, we only need to verify the desired assertion for  $x \in \mathbb{R}^d$  with  $|x| > \frac{\kappa(1-5\varepsilon)(1-4\varepsilon)}{\varepsilon}$ . According to (3.33), we have for any  $x \in \mathbb{R}^d$  with  $|x| > \frac{\kappa(1-5\varepsilon)(1-4\varepsilon)}{\varepsilon}$  that

$$\begin{aligned} \exp\left(-\frac{1}{\kappa(1-6\varepsilon)}|x| \log(1+|x| + \sup_{|z| \leq |x|+2\varepsilon\kappa} V(z)) - C(\kappa, \varepsilon)\right) \\ \leq T_{t_0}^V(\mathbf{1}_D)(x) \leq cT_{t_0}^V(\phi_1)(x) = ce^{-\lambda_1 t_0} \phi_1(x), \end{aligned}$$

where  $c := (\inf_{y \in D} \phi_1(y))^{-1} < \infty$ . This immediately yields the desired assertion.  $\square$

**3.2. Intrinsic local super Poincaré inequality.** In this part, we will present the following local intrinsic super Poincaré inequality.

**Proposition 3.5.** *Let  $\varphi$  be a positive and continuous function on  $\mathbb{R}^d$ . For any  $r \geq \kappa$ ,  $s > 0$  and  $f \in C_c^2(\mathbb{R}^d)$ ,*

$$\begin{aligned} \int_{B(0,r)} f^2(x) dx \leq sD^V(f, f) \\ + \frac{c(\kappa)}{\inf_{|x| \leq r+\kappa} \varphi^2(x)} (1 + s^{-\frac{d}{\alpha_1}}) \left( \int_{B(0,r+\kappa)} |f(x)| \varphi(x) dx \right)^2. \end{aligned}$$

*Proof.* (i) According to (1.2) and the fact that  $V \geq 0$ , for any  $f \in C_c^2(\mathbb{R}^d)$ ,

$$\begin{aligned} (3.38) \quad D_{\alpha_1, \kappa}(f, f) &:= c_1 \iint_{\{|x-y| \leq \kappa\}} (f(x) - f(y))^2 |x - y|^{-d-\alpha_1} dx dy \\ &\leq D^V(f, f). \end{aligned}$$

Next, we follow the argument of [5, Theorem 3.1] to obtain that for any  $s > 0$ ,  $r \geq \kappa$  and  $f \in C_c^2(\mathbb{R}^d)$ ,

$$(3.39) \quad \int_{B(0,r)} f^2(x) dx \leq sD_{\alpha_1, \kappa}(f, f) + C(\kappa) (1 + s^{-\frac{d}{\alpha_1}}) \left( \int_{B(0,r+\kappa)} |f(x)| dx \right)^2$$

holds with some constant  $C(\kappa) > 0$ . If (3.39) holds, then, combining it with (3.38) above will complete the proof.

(ii) Next, we turn to the proof of (3.39). For any  $0 < s \leq r$  and  $f \in C_c^2(\mathbb{R}^d)$ , define

$$f_s(x) := \frac{1}{|B(0, s)|} \int_{B(x, s)} f(z) dz, \quad x \in B(0, r).$$

We have

$$\sup_{x \in B(0, r)} |f_s(x)| \leq \frac{1}{|B(0, s)|} \int_{B(0, r+s)} |f(z)| dz,$$

and

$$\begin{aligned} \int_{B(0, r)} |f_s(x)| dx &\leq \int_{B(0, r)} \frac{1}{|B(0, s)|} \int_{B(x, s)} |f(z)| dz dx \\ &\leq \int_{B(0, r+s)} \left( \frac{1}{|B(0, s)|} \int_{B(z, s)} dx \right) |f(z)| dz \\ &\leq \int_{B(0, r+s)} |f(z)| dz. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{B(0,r)} f_s^2(x) dx &\leq \left( \sup_{x \in B(0,r)} |f_s(x)| \right) \int_{B(0,r)} |f_s(x)| dx \\ &\leq \frac{1}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(z)| dz \right)^2. \end{aligned}$$

Therefore, for any  $f \in C_c^2(\mathbb{R}^d)$  and  $0 < s \leq r$ ,

$$\begin{aligned} \int_{B(0,r)} f^2(x) dx &\leq 2 \int_{B(0,r)} (f(x) - f_s(x))^2 dx + 2 \int_{B(0,r)} f_s^2(x) dx \\ &\leq 2 \int_{B(0,r)} \frac{1}{|B(0,s)|} \int_{B(x,s)} (f(x) - f(y))^2 dx dy \\ &\quad + \frac{2}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(z)| dz \right)^2 \\ &\leq \left( \frac{2s^{d+\alpha_1}}{|B(0,s)|} \right) \int \int_{\{|x-y| \leq s\}} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha_1}} dx dy \\ &\quad + \frac{2}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(z)| dz \right)^2. \end{aligned}$$

In particular, for any  $f \in C_c^2(\mathbb{R}^d)$  and  $0 < s \leq \kappa \leq r$ ,

$$\int_{B(0,r)} f^2(x) dx \leq c_3 \left[ s^{\alpha_1} D_{\alpha_1, \kappa}(f, f) + s^{-d} \left( \int_{B(0, r+\kappa)} |f(z)| dz \right)^2 \right],$$

which implies that there exists a constant  $s_0 := s_0(\kappa) > 0$  such that for all  $s \in (0, s_0]$ ,

$$\begin{aligned} \int_{B(0,r)} f^2(x) dx &\leq s D_{\alpha_1, \kappa}(f, f) \\ &\quad + c_4 s^{-d/\alpha_1} \left( \int_{B(0, r+\kappa)} |f(z)| dz \right)^2, \quad r \geq \kappa, f \in C_c^2(\mathbb{R}^d). \end{aligned}$$

This proves the desired assertion (3.39).  $\square$

**Remark 3.6.** One also can derive the inequality (3.39) along the lines of the proof of [3, Lemma 2.1]. Indeed, when  $\kappa = 1$ , the inequality (3.39) is just [3, Lemma 2.1]. For general  $\kappa > 0$ , the proof of (3.39) is almost the same as that of [3, Lemma 2.1], and one only need to replace  $B(0, \frac{1}{2})$  in [3, (2.17)] by  $B(0, \frac{\kappa}{2})$ .

### 3.3. Sobolev inequalities.

**Proposition 3.7.** *If  $d > \alpha_1$ , then there exists a constant  $c_0(\kappa) > 0$  such that the following Sobolev inequality holds*

$$(3.40) \quad \|f\|_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d; dx)}^2 \leq c_0(\kappa) \left[ D(f, f) + \|f\|_{L^2(\mathbb{R}^d; dx)}^2 \right], \quad f \in C_c^\infty(\mathbb{R}^d).$$

*Proof.* According to [4, (2.3)] and (1.2), we get that for any  $f \in C_c^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} \|f\|_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d; dx)}^2 &\leq c_5 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha_1}} dx dy \\ &\leq c_5 \int \int_{\{|x-y| \leq \kappa\}} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha_1}} dx dy + c_6(\kappa) \|f\|_{L^2(\mathbb{R}^d; dx)}^2 \\ &\leq c_1 c_5 D(f, f) + c_6(\kappa) \|f\|_{L^2(\mathbb{R}^d; dx)}^2. \end{aligned}$$

This completes the proof.  $\square$

**3.4. General results about intrinsic ultracontractivity of Feynman-Kac semigroups.** According to Propositions 3.4, 3.5, 3.7 and Theorem 2.1, we immediately have the following statement.

**Theorem 3.8.** *Let  $(T_t^V)_{t \geq 0}$  be a compact Feynman-Kac semigroup on  $L^2(\mathbb{R}^d, dx)$  given in the beginning of this section (or in Subsection 1.1), and  $V$  be a locally bounded non-negative measurable function on  $\mathbb{R}^d$  such that assumptions **(A2)** and **(A4)** hold. For some  $\varepsilon \in (0, \min(1/11, \varepsilon_0))$ , define*

$$(3.41) \quad \varphi(x) := \exp \left( - \frac{1}{\kappa(1 - 6\varepsilon)} |x| \log(1 + |x| + \sup_{|z| \leq |x| + 2\varepsilon\kappa} V(z)) \right),$$

and

$$\alpha(r, s) := \frac{c(\kappa)}{\inf_{|x| \leq r + \kappa} \varphi^2(x)} \left( 1 + s^{-\frac{d}{\alpha_1}} \right),$$

where  $\varepsilon_0$  is the constant in Lemma 3.2 and  $c(\kappa)$  is the constant in Proposition 3.5. For the constant  $K$  in **(A4)**, let

$$\begin{aligned} \Phi(R) &:= \inf_{x \in \mathbb{R}^d; |x| \geq R, V(x) > K} V(x), \\ \Theta(R) &:= |\{x \in \mathbb{R}^d : |x| \geq R, V(x) \leq K\}|, \\ \Psi(R) &:= \frac{1}{\Phi(R)} + c_0(\kappa) \Theta(R)^{\alpha_1/d}, \end{aligned}$$

where  $c_0(\kappa)$  is a constant in (3.40). We furthermore define

$$(3.42) \quad \begin{aligned} \beta_\Phi(s) &:= \left[ 1 + \alpha \left( \Phi^{-1} \left( \frac{2}{s} \right), \frac{s}{2} \right) \right], \\ \beta_\Psi(s) &:= \left[ 1 + \alpha \left( \Psi^{-1} \left( \frac{s}{4} \right), \frac{s}{4} \right) \right], \\ \gamma(s) &:= \Theta^{-1} \left( \frac{s}{2} \right). \end{aligned}$$

(1) If  $\lim_{|x| \rightarrow \infty} V(x) = \infty$ , and

$$\int_t^\infty \frac{\beta_\Phi^{-1}(s)}{s} ds < \infty, \quad t \gg 1,$$

then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

(2) If  $d > \alpha_1$  and

$$\int_t^\infty \frac{\beta_\Psi^{-1}(s)}{s} ds < \infty, \quad t \gg 1,$$

then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

(3) Suppose that there exists a constant  $\delta > 1$  such that

$$(3.43) \quad \sum_{n=1}^{\infty} \gamma(s_n) \delta^n < \infty$$

and

$$(3.44) \quad \int_t^{\infty} \frac{\tilde{\beta}_{\Phi}^{-1}(s)}{s} ds < \infty, \quad t \gg 1,$$

where  $s_n := \beta_{\Phi}^{-1}(\frac{c_1 \delta^n}{2})$  with  $c_1 := \|\phi_1\|_{\infty}^2$ , and

$$(3.45) \quad \tilde{\beta}_{\Phi}(s) := 2\beta_{\Phi} \left( \gamma^{-1} \left( \frac{1}{4\delta^{n_0(s)+1}} \right) \right)$$

with

$$n_0(s) := \inf \left\{ N \geq 1 : \frac{4\delta(\sqrt{\delta} + 1)s_N}{\sqrt{\delta} - 1} + 2\gamma^{-1} \left( \frac{1}{4\delta^{N+1}} \right) \leq s \right\}.$$

Then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

#### 4. PROOFS OF THEOREMS AND EXAMPLES

In the section, we will give the proofs of all the statements in Section 1. First, we present the

*Proof of Theorem 1.3.* (1) It is clear that

$$\lim_{|x| \rightarrow \infty} V(x) = \infty.$$

Let  $\varepsilon \in (0, \min(1/11, \varepsilon_0))$ . Since

$$V(x) \leq c_4 |x|^{\theta_3} \log^{\theta_4}(1 + |x|), \quad x \in \mathbb{R}^d,$$

we have the following estimate for the function  $\varphi$  given by (3.41)

$$\varphi(x) \geq \exp \left( - \frac{\theta_3}{\kappa(1 - 7\varepsilon)} (1 + |x|) \log(1 + |x|) - C(\kappa, \varepsilon, \theta_3, \theta_4) \right).$$

On the other hand, since

$$V(x) \geq c_3 |x|^{\theta_1} \log^{\theta_2}(1 + |x|), \quad x \in \mathbb{R}^d,$$

we obtain

$$\Phi(r) \geq c_3 r^{\theta_1} \log^{\theta_2}(1 + r).$$

Therefore, the rate function  $\beta_{\Phi}(r)$  defined by (3.42) satisfies that for  $s > 0$  small enough

$$\beta_{\Phi}(s) \leq C(\kappa, \varepsilon) \exp \left\{ C(\kappa, \varepsilon, \theta_3, \theta_4) \left( 1 + s^{-\frac{1}{\theta_1}} \log^{1-\frac{\theta_2}{\theta_1}}(1 + s^{-1}) \right) \right\}.$$

In particular,

$$\beta_{\Phi}^{-1}(r) \leq \frac{C}{\log^{\theta_1}(1 + r) \log^{\theta_2 - \theta_1}(e + r)}, \quad r > 0 \text{ large enough.}$$

Then, if  $\theta_1 = 1$  and  $\theta_2 > 2$  or if  $\theta_1 > 1$ , the intrinsic ultracontractivity of  $(T_t^V)_{t \geq 0}$  immediately follows from Theorem 3.8(1).

(2) The required lower bound for the ground state  $\phi_1$  immediately follows from (3.37). Next, we will verify the upper bound. If  $\theta_1 = 1$  and  $\theta_2 > 2$  or if  $\theta_2 > 1$ , then the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive. For any  $0 < \lambda < \theta_1$ , let

$$\psi(x) := \exp \left( -\frac{\lambda}{2\kappa} \sqrt{1+|x|^2} \log(1+|x|^2) \right).$$

Suppose that assumption **(A4)** holds and for any  $x, y \in \mathbb{R}^d$  with  $|x - y| > \kappa$ ,  $J(x, y) = 0$ . Then, the generator  $L^V$  of the associated Feynman-Kac semigroup  $(T_t^V)_{t \geq 0}$  enjoys the expression (1.4). By the approximation argument, it is easy to verify that  $\psi \in \mathcal{D}(L^V)$ . For  $|x|$  large enough, we obtain by the mean value theorem that

$$\begin{aligned} L^V \psi(x) &\leq C_1 \sup_{z \in B(x, \kappa)} \left( |\nabla \psi(z)| + |\nabla^2 \psi(z)| \right) - V(x) \psi(x) \\ &\leq C_2(\kappa, \lambda) \log^2(1+|x|) \cdot \exp \left( -\frac{\lambda}{\kappa} (|x| - \kappa) \log(1+|x| - \kappa) - C_3(\kappa, \lambda) \right) \\ &\quad - c_3(1+|x|)^{\theta_1} \log^{\theta_2}(1+|x|) \psi(x) \\ &\leq C_4(\kappa, \lambda) (1+|x|)^\lambda \log^2(1+|x|) \psi(x) - c_3(1+|x|)^{\theta_1} \log^{\theta_2}(1+|x|) \psi(x). \end{aligned}$$

Since  $0 < \lambda < \theta_1$ ,

$$L^V \psi(x) \leq 0$$

for  $|x|$  large enough. Note that the function  $x \mapsto L^V \psi(x)$  is locally bounded, we know from the inequality above that (2.30) holds with some constant  $\lambda > 0$ . Therefore, the required upper bound for  $\phi_1$  follows from Proposition 2.3.  $\square$

Indeed, according to Theorem 3.8(1), we also have the following statements. The proofs are similar to that of Theorem 1.3, and so we omit them here.

**Proposition 4.1.** *Suppose that (1.2), (1.3), assumptions **(A1)** and **(A2)** hold. Then, we have the following two assertions.*

- (1) *If there are positive constants  $c_5, c_6, \theta_5, \theta_6$  with  $\theta_5 > \theta_6 + 1$  such that for all  $x \in \mathbb{R}^d$ ,*

$$c_5(1+|x|^{\theta_5}) \leq V(x) \leq e^{c_6(1+|x|^{\theta_6})},$$

*then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive, and for any  $\varepsilon > 0$  there is a constant  $C_3 := C_3(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$C_3 \exp \left( -\frac{(1+\varepsilon)}{\kappa} |x|^{\theta_6+1} \right) \leq \phi_1(x).$$

*Additionally, if moreover **(A3)** also holds and*

$$J(x, y) = 0, \quad x, y \in \mathbb{R}^d \text{ with } |x - y| > \kappa,$$

*then for any  $\varepsilon > 0$ , there exists a constant  $C_4 := C_4(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$\phi_1(x) \leq C_4 \exp \left( -\frac{(1-\varepsilon)\theta_5}{\kappa} |x| \log(1+|x|) \right).$$

- (2) *If there are positive constants  $c_7, c_8, \theta_7, \theta_8$  with  $\theta_7 \leq \theta_8$  such that for all  $x \in \mathbb{R}^d$ ,*

$$e^{c_7(1+|x|^{\theta_7})} \leq V(x) \leq e^{c_8(1+|x|^{\theta_8})},$$

then  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive, and for any  $\varepsilon > 0$  there is a constant  $C_5 := C_5(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,

$$C_5 \exp \left( - \frac{c_8(1+\varepsilon)}{\kappa} |x|^{\theta_8+1} \right) \leq \phi_1(x).$$

Additionally, if moreover **(A3)** also holds and

$$J(x, y) = 0, \quad x, y \in \mathbb{R}^d \text{ with } |x - y| > \kappa,$$

then for any  $\frac{\theta_7}{\theta_7+1} < \varepsilon < 1$ , there exists a constant  $C_6 := C_6(\varepsilon) > 0$  such that for all  $x \in \mathbb{R}^d$ ,

$$\phi_1(x) \leq C_6 \exp \left( - \frac{c_7(1-\varepsilon)}{\kappa} |x|^{\theta_7+1} \right).$$

Next, we turn to the

*Proof of Example 1.4.* (1) According to Theorem 1.3, we know that  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive if  $V(x) = |x|^\theta$  for some  $\theta > 1$ . Now we are going to verify that if  $V(x) = |x|^\theta$  for some  $0 < \theta \leq 1$ , then  $(T_t^V)_{t \geq 0}$  is not intrinsically ultracontractive. We mainly use the method of [15, Theorem 1.6] (see [15, pp. 5055-5056]) and disprove [15, Condition 1.3, p. 5027]. In fact, according to [15, Condition 1.3, p. 5027], if  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive, then for every fixed  $t \in (0, 1]$ , there exists a constant  $C_t > 0$  such that

$$(4.46) \quad T_t^V(\mathbf{1}_D)(x) \geq C_t T_t^V(\mathbf{1}_{B(x,1)})(x).$$

Let  $p(t, x, y)$  be the heat kernel for the associated process  $(X_t)_{t \geq 0}$ . According to [7, (1.16) in Theorem 1.2 and (1.20) in Theorem 1.4], for any fixed  $t \in (0, 1]$  and  $|x - y|$  large enough,

$$p(t, x, y) \leq C_1 t \exp \left( - C_2 |x - y| \left( \log \frac{|x - y|}{t} \right)^{\frac{\gamma-1}{\gamma}} \right).$$

Set  $D = B(0, 1)$ . For  $|x|$  large enough,

$$T_t^V(\mathbf{1}_D)(x) \leq \int_D p(t, x, y) dy \leq C_3 t \exp \left( - C_2 (|x| - 1) \left( \log \frac{|x| - 1}{t} \right)^{\frac{\gamma-1}{\gamma}} \right).$$

On the other hand, for  $|x|$  large enough,

$$\begin{aligned} T_t^V(\mathbf{1}_{B(x,1)})(x) &\geq \mathbb{E}^x \left( \tau_{B(x,1)} > t; \exp \left( - \int_0^t V(X_s) ds \right) \right) \\ &\geq C \mathbb{P}^x(\tau_{B(x,1)} > t) e^{-t|x|^\theta} \\ &\geq C \mathbb{P}^x(\tau_{B(x,1)} > 1) e^{-t|x|^\theta} \\ &\geq C e^{-t|x|^\theta}. \end{aligned}$$

Combining both conclusions above with the fact that  $\theta \in (0, 1]$ , we get that for any fixed  $t \in (0, 1]$ , the inequality (4.46) does not hold for any constant  $C_t > 0$ , which contradicts with [15, Condition 1.3, p. 5027]. Hence, according to the remark below [15, Condition 1.3, p. 5027], the semigroup  $(T_t^V)_{t \geq 0}$  is not intrinsically ultracontractive.

(2) If  $\gamma = \infty$  and  $\theta > 1$ , then the ground state estimate (1.9) immediately follows from Theorem 1.3. When  $1 < \gamma < \infty$  and  $\theta > 1$ , one can apply Proposition 3.4 to



get a lower bound estimate for  $\phi_1$ , which however is not optimal. Instead, we will adopt a slightly different argument from that of Proposition 3.4, and will derive a more accurate lower bound estimate, which is partly inspired by [7, Theorem 5.4].

For any  $\lambda > 0$ , we choose a constant

$$0 < \varepsilon < \varepsilon_0 \wedge \left( \frac{1}{2} \theta^{\frac{1}{\gamma}} \left( (1 + \lambda)^{\frac{1}{\gamma}} - 1 \right) \right),$$

where  $\varepsilon_0 > 0$  is the same constant in Lemma 3.2. For every  $x \in \mathbb{R}^d$  with

$$|x| \geq e^{2^\gamma \theta(1+2\varepsilon)^\gamma} \vee (e - 1) \quad \text{and} \quad \theta^{-\frac{1}{\gamma}} |x| \log^{-\frac{1}{\gamma}}(1 + |x|) \geq 1,$$

let

$$n = \left\lfloor \theta^{-\frac{1}{\gamma}} |x| \log^{-\frac{1}{\gamma}}(1 + |x|) \right\rfloor + 1,$$

and  $x_i := ix/n$  for any  $0 \leq i \leq n$ , where  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ . Next, for all  $0 \leq i \leq n$ , we set  $D_i := B(x_i, \varepsilon)$  and  $\tilde{D}_i := B(x_i, \varepsilon/2)$ . Note that

$$(4.47) \quad \theta^{-\frac{1}{\gamma}} |x| \log^{-\frac{1}{\gamma}}(1 + |x|) \leq n \leq \theta^{-\frac{1}{\gamma}} |x| \log^{-\frac{1}{\gamma}}(1 + |x|) + 1,$$

we can check that for each  $0 \leq i \leq n - 1$ ,

$$1 \leq \frac{1}{2} \theta^{\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}(1 + |x|) - 2\varepsilon \leq \frac{1}{\theta^{-\frac{1}{\gamma}} \log^{-\frac{1}{\gamma}}(1 + |x|) + 1} - 2\varepsilon \leq \text{dist}(D_i, D_{i+1})$$

and for every  $z_i \in D_i$  and  $z_{i+1} \in D_{i+1}$ ,

$$|z_i - z_{i+1}| \leq \frac{|x|}{n} + 2\varepsilon \leq \theta^{\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}(1 + |x|) + 2\varepsilon \leq \left( (1 + \lambda) \theta \log(1 + |x|) \right)^{\frac{1}{\gamma}}.$$

In the following, we define for all  $n \geq 1$

$$\begin{aligned} \tilde{\tau}_{D_i} &:= \inf\{t \geq \tilde{\tau}_{D_{i+1}} : X_t \notin D_i\}, \quad 1 \leq i \leq n - 1; \\ \tilde{\tau}_{D_n} &:= \tau_{D_n}. \end{aligned}$$

By the convention, we also set  $\tilde{\tau}_{D_{n+1}} = 0$ . Let  $T(1, \varepsilon)$  be the same constant in Lemma 3.2 with  $\kappa = 1$ .

First, if  $X_{\tilde{\tau}_{D_i}} \in \tilde{D}_{i-1}$ , then we have for each  $i \geq 2$ ,  $j \geq 1$  and  $t_0 = T(1, \varepsilon)$ ,

$$\begin{aligned} & \mathbb{P}^{X_{\tilde{\tau}_{D_i}}} \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2} \right) \\ & \geq \inf_{x \in \tilde{D}_{i-1}} \mathbb{P}^x \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2} \right) \\ & = \inf_{x \in \tilde{D}_{i-1}} \int_{\frac{t_0}{(j+1)n}}^{\frac{t_0}{jn}} \int_{D_{i-1}} p_{D_{i-1}}(s, x, y) \int_{\tilde{D}_{i-2}} J(y, z) dz dy ds \\ & \geq C \inf_{x \in \tilde{D}_{i-1}} \int_{\frac{t_0}{(j+1)n}}^{\frac{t_0}{jn}} \int_{D_{i-1}} p_{D_{i-1}}(s, x, y) \int_{\tilde{D}_{i-2}} e^{-|y-z|^\gamma} dz dy ds \\ & \geq C \inf_{x \in \tilde{D}_{i-1}} \int_{\frac{t_0}{(j+1)n}}^{\frac{t_0}{jn}} \mathbb{P}^x(\tau_{D_{i-1}} > s) ds e^{-(1+\lambda)(\theta \log(1+|x|))} \\ & \geq \frac{C t_0}{j(j+1)n(1+|x|)^{(1+\lambda)\theta}} \inf_{x \in \tilde{D}_{i-1}} \mathbb{P}^x \left( \tau_{D_{i-1}} > \frac{t_0}{jn} \right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{Ct_0}{j(j+1)n(1+|x|)^{(1+\lambda)\theta}} \inf_{x \in \tilde{D}_{i-1}} \mathbb{P}^x \left( \tau_{B(x, \varepsilon/2)} > t_0 \right) \\
&\geq \frac{C}{j(j+1)n(1+|x|)^{(1+\lambda)\theta}},
\end{aligned}$$

where the equality above is due to (3.1), in the third inequality we have used the fact that  $|y - z| \leq ((1 + \lambda)\theta \log |x|)^{1/\gamma}$  for  $y \in D_{i-1}$  and  $z \in \tilde{D}_{i-2}$ , and the last inequality follows from Lemma 3.1.

Hence, if  $X_{\tilde{\tau}_{D_i}} \in \tilde{D}_{i-1}$ , then we have for all  $i \geq 2$ ,

$$\begin{aligned}
&\mathbb{E}^{X_{\tilde{\tau}_{D_i}}} \left( 0 < \tau_{D_{i-1}} < \frac{t_0}{n}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2}; \exp \left( - \int_0^{\tau_{D_{i-1}}} V(X_s) ds \right) \right) \\
&\geq \sum_{j=1}^{\infty} \mathbb{E}^{X_{\tau_{D_i}}} \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2}; \exp \left( - \int_0^{\tau_{D_{i-1}}} V(X_s) ds \right) \right) \\
&\geq \sum_{j=1}^{\infty} \exp \left( - \frac{t_0}{jn} \sup_{x \in D_{i-1}} V(x) \right) \inf_{y \in \tilde{D}_{i-1}} \mathbb{P}^y \left( \frac{t_0}{(j+1)n} \leq \tau_{D_{i-1}} < \frac{t_0}{jn}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2} \right) \\
&\geq \frac{C}{n(1+|x|)^{(1+\lambda)\theta}} \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \exp \left( - \frac{t_0}{jn} \sup_{x \in D_{i-1}} V(x) \right) \\
&\geq \frac{C}{(1+|x|)^{(1+\lambda)\theta} (n + \sup_{x \in D_{i-1}} V(x))},
\end{aligned}$$

where in the last inequality we have used (3.35).

Furthermore, we find that (3.34) and (3.36) are still valid here. Therefore, combining with all the estimates above, we obtain that for  $|x|$  large enough

$$\begin{aligned}
T_{t_0}^V(\mathbf{1}_D)(x) &\geq C_4 \prod_{i=1}^n \left( \frac{C_5}{|x|^{(1+\lambda)\theta} (n + \sup_{z \in D_{i-1}} V(z))} \right) \\
&\geq C_4 \left( \frac{C_5}{|x|^{(1+\lambda)\theta} (n + \sup_{|z| \leq |x| + \varepsilon} V(z))} \right)^n \\
&\geq \exp \left( - (1 + \lambda) |x| (\theta \log |x|)^{-\frac{1}{\gamma}} \log(1 + |x|^{(2+\lambda)\theta}) - C(\theta) \right) \\
&\geq \exp \left( - (1 + \lambda)(2 + \lambda) \theta^{\frac{\gamma-1}{\gamma}} |x| (\log |x|)^{\frac{\gamma-1}{\gamma}} - C(\theta)(1 + |x|) \right),
\end{aligned}$$

where in the third inequality we have used the property (4.47).

Hence, following the same argument as that of Proposition 3.4, we finally arrive at

$$\phi_1(x) \geq \exp \left( - (1 + \lambda)(2 + \lambda) \theta^{\frac{\gamma-1}{\gamma}} |x| \log^{\frac{\gamma-1}{\gamma}}(1 + |x|) - C(\theta)(1 + |x|) \right).$$

In particular, by taking  $\lambda > 0$  small enough in the inequality above, we indeed can get the lower bound estimate in (1.10) with any constant  $c_4 > 2$ .

(3) Let

$$\psi(x) := \exp \left( - (c_0 \theta)^{\frac{\gamma-1}{\gamma}} \sqrt{1 + |x|^2} \log^{\frac{\gamma-1}{\gamma}} \sqrt{1 + |x|^2} \right),$$

where  $c_0 > 0$  is a constant to be determined later.

Under assumption **(A3)**, by the approximation argument again it is easy to verify that  $\psi \in \mathcal{D}(L^V)$ , we know from (1.4) that

$$\begin{aligned} L^V \psi(x) &= \int_{\mathbb{R}^d} \left( \psi(x+z) - \psi(x) - \langle \nabla \psi(x), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) J(x, x+z) dz \\ &\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} \langle \nabla \psi(x), z \rangle (J(x, x+z) - J(x, x-z)) dz - V(x)\psi(x) \\ &\leq c_1 \sup_{z \in B(x,1)} (|\nabla^2 \psi(x)| + |\nabla \psi(x)|) \\ &\quad + \int_{\{|z| > 1\}} (\psi(x+z) - \psi(x)) J(x, x+z) dz - V(x)\psi(x) \\ &=: I_1(x) + I_2(x) - V(x)\psi(x). \end{aligned}$$

According to the proof of Theorem 1.3 and the mean value theorem, for  $|x|$  large enough,

$$\begin{aligned} I_1(x) &\leq C(\theta) \log^{\frac{2(\gamma-1)}{\gamma}}(1+|x|) \exp \left( - (c_0 \theta)^{\frac{\gamma-1}{\gamma}} (|x|-1) \log^{\frac{\gamma-1}{\gamma}}(|x|-1) \right) \\ &\leq C(\theta) \exp \left( (c_0 \theta)^{\frac{\gamma-1}{\gamma}} \log^{\frac{\gamma-1}{\gamma}}(1+|x|) \right) \log^{\frac{2(\gamma-1)}{\gamma}}(1+|x|) \psi(x). \end{aligned}$$

On the other hand,

$$I_2(x) \leq \int_{\{|z| > 1\}} \psi(x+z) J(x, x+z) dz \leq c_2 \int_{\{|z| > 1\}} \psi(x+z) e^{-|z|^\gamma} dz.$$

For each  $i \geq 1$ , set  $A_i := B(0, i+1) \setminus B(0, i)$ . Since for every  $x \in \mathbb{R}^d$  and  $z \in A_i$ ,

$$|x| - i - 1 \leq |x+z| \leq |x| + i + 1,$$

we get that for any  $0 < \lambda < 1$  and any  $x \in \mathbb{R}^d$  with  $|x|$  large enough,

$$\begin{aligned} &\int_{\{|z| > 1\}} \psi(x+z) e^{-|z|^\gamma} dz \\ &= \sum_{i=1}^{\infty} \int_{A_i} \psi(x+z) e^{-|z|^\gamma} dz \\ &\leq c_3 \sum_{i=2}^{\infty} i^{d-1} e^{-(i-1)^\gamma} \sup_{|x|-i-1 \leq |z| \leq |x|+i+1} |\psi(z)| \\ &\leq c_4 \psi(x) \sum_{i=2}^{\infty} i^{d-1} \exp \left[ \left( (1+\lambda) c_0 \theta \log |x| \right)^{\frac{\gamma-1}{\gamma}} i - (i-1)^\gamma \right], \end{aligned}$$

where in the first inequality we have used the fact that  $|A_i| \leq C i^{d-1}$ , and the second inequality follows from the fact that for  $|x|$  large enough

$$\sup_{|z| \geq |x|-i-1} |\psi(z)| \leq \exp \left[ \left( (1+\lambda) c_0 \theta \log |x| \right)^{\frac{\gamma-1}{\gamma}} i \right] \psi(x),$$

thanks to again the mean value theorem.

Furthermore, set  $N(x) := \left\lfloor (6(1+\lambda) c_0 \theta \log |x|)^{\frac{1}{\gamma}} \right\rfloor + 1$ . Noticing that for every  $i > N(x)$ ,

$$\left( (1+\lambda) c_0 \theta \log |x| \right)^{\frac{\gamma-1}{\gamma}} i - (i-1)^\gamma \leq -\frac{i^\gamma}{2},$$

we have

$$\begin{aligned}
& \sum_{i=2}^{\infty} i^{d-1} \exp \left[ ((1+\lambda) c_0 \theta \log |x|)^{\frac{\gamma-1}{\gamma}} i - (i-1)^\gamma \right] \\
&= \sum_{i=2}^{N(x)} i^{d-1} \exp \left[ ((1+\lambda) c_0 \theta \log |x|)^{\frac{\gamma-1}{\gamma}} i - (i-1)^\gamma \right] \\
&\quad + \sum_{i=N(x)+1}^{\infty} i^{d-1} \exp \left[ ((1+\lambda) c_0 \theta \log |x|)^{\frac{\gamma-1}{\gamma}} i - (i-1)^\gamma \right] \\
&\leq N(x)^d \exp \left[ \sup_{s \in \mathbb{R}} \left( ((1+\lambda) c_0 \theta \log |x|)^{\frac{\gamma-1}{\gamma}} s - (s-1)^\gamma \right) \right] + \sum_{i=N(x)+1}^{\infty} i^{d-1} e^{-\frac{i^\gamma}{2}} \\
&\leq C(\theta) (\theta \log |x|)^{\frac{d}{\gamma}} \exp \left( (1+2\lambda) c_0 \theta \left( \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}} - \left( \frac{1}{\gamma} \right)^{\frac{\gamma}{\gamma-1}} \right) \log |x| \right) \\
&\quad + \exp \left[ -\frac{(1+\lambda) c_0 \theta}{4} \log |x| \right],
\end{aligned}$$

where in the last inequality we have used the facts that for  $|x|$  large enough,

$$\begin{aligned}
& \sup_{s \in \mathbb{R}} \left\{ ((1+\lambda) c_0 \theta \log |x|)^{\frac{\gamma-1}{\gamma}} s - (s-1)^\gamma \right\} \\
&\leq (1+2\lambda) c_0 \theta \left( \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}} - \left( \frac{1}{\gamma} \right)^{\frac{\gamma}{\gamma-1}} \right) \log |x|
\end{aligned}$$

and

$$\sum_{i=n}^{\infty} i^{d-1} e^{-\frac{i^\gamma}{2}} \leq e^{-\frac{n^\gamma}{4}}$$

for  $n$  large enough.

Combining all the estimates above and taking

$$c_0 = \frac{1}{2(1+2\lambda) \left( \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}} - \left( \frac{1}{\gamma} \right)^{\frac{\gamma}{\gamma-1}} \right)},$$

we derive that for  $|x|$  large enough,

$$L^V(x) \psi(x) \leq C(\theta) |x|^{\frac{2}{3}\theta} \psi(x) - V(x) \psi(x) \leq 0,$$

which implies that

$$L^V \psi(x) \leq C(\theta) \psi(x), \quad x \in \mathbb{R}^d$$

for some constant  $C(\theta) > 0$ . Therefore, according to Proposition 2.3, we can obtain the desired upper bound estimate for  $\phi_1$  as that in (1.10).  $\square$

At the last, we turn to the proofs of Theorem 1.6 and Example 1.7.

*Proof of Theorem 1.6.* (1) Following the argument of Theorem 1.3, for  $R, r > 0$  large enough

$$\Phi(R) \geq CR \log^{\theta_1} R$$

and

$$\beta_{\Phi}^{-1}(r) \leq \frac{C}{\log(1+r) \log^{\theta_1-1} \log(e+r)}.$$

Hence, for  $s > 0$  small enough,

$$\gamma(s) = \Theta \left( \Phi^{-1} \left( \frac{2}{s} \right) \right) \leq C_1 \exp \left( -C_2 c_6 \left( \frac{1}{s} \right)^{\eta_1} \left( \log \frac{1}{s} \right)^{\eta_2 - \eta_1 \theta_1} \right).$$

Then, for any fixed  $\delta > e$ , there is an integer  $N_0(\delta) \geq 1$  such that for all  $n \geq N_0(\delta)$ ,

$$s_n := \beta_{\Phi}^{-1}(c\delta^n) \leq C_3 (\log \delta)^{-1} n^{-1} (\log n)^{-(\theta_1-1)},$$

where  $C_3 > 0$  is independent of  $\delta$ . Therefore, for  $n$  large enough,

$$(4.48) \quad \gamma(s_n) \leq C \exp \left[ -c_6 C_4 (\log \delta)^{\eta_1} n^{\eta_1} (\log n)^{\eta_2 - \eta_1} \right],$$

where  $C_4$  is a constant independent of  $\delta$ .

According to (4.48), if  $\eta_1 = 1$  and  $\eta_2 > 1$ , then (3.43) holds true, and we have the following estimate for the rate function  $\tilde{\beta}_{\Phi}(s)$  defined by (3.45)

$$(4.49) \quad \tilde{\beta}_{\Phi}(s) \leq C \exp \left[ C \left( 1 + \frac{1}{s} \right) \left( \log^{1-\theta_1} \left( 1 + \frac{1}{s} \right) \right) \right],$$

which implies that (3.44) is satisfied. Therefore, according to Theorem 3.8(3), we know that the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive.

On the other hand, it follows from (4.48) that, when  $\eta_1 = \eta_2 = 1$  and  $c_6 > \frac{1}{C_4}$ , (3.43) holds true. Then, following the arguments above, we can get the same estimate (4.49) for  $\tilde{\beta}_{\Phi}(s)$  (possibly with different constant  $C$  in (4.49)), and so the semigroup  $(T_t^V)_{t \geq 0}$  is intrinsically ultracontractive. The proof of the first assertions is complete.

(2) When  $d > \alpha_1$ , there exists a constant  $C > 0$  such that for  $R$  large enough

$$\Psi(R) \leq C \left( \frac{1}{R \log^{\theta_1} R} + \frac{1}{R \log^{\alpha_1 \eta_3 / d} R} \right).$$

Then, following the same argument of part (1), we can arrive at the second conclusion by Theorem 3.8(2).  $\square$

*Proof of Example 1.7.* Here we still try to disprove [15, Condition 1.3, p. 5027]. Let  $D = B(0, 1)$  and  $t = 1$ . According to the proof of Example 1.4, for all  $n$  large enough,

$$(4.50) \quad T_1^V(\mathbf{1}_D)(x_n) \leq C_1 \exp \left( -C_2 |x_n| \log |x_n| \right) = C_1 \exp \left( -C_3 n^{k_0} \log n \right).$$

On the other hand, for  $n$  large enough

$$\begin{aligned} T_1^V(\mathbf{1}_{B(x_n, 1)})(x_n) &\geq T_1^V(\mathbf{1}_{B(x_n, r_n)})(x_n) \\ &\geq \mathbb{E}^{x_n} \left( \tau_{B(x_n, r_n)} > 1; \exp \left( - \int_0^1 V(X_s) ds \right) \right) \\ &= e^{-1} \mathbb{P}^{x_n}(\tau_{B(x_n, r_n)} > 1) \\ &= e^{-1} \mathbb{P}^0(\tau_{B(0, r_n)} > 1), \end{aligned}$$

where the first equality follows from the fact that  $V(z) = 1$  for every  $z \in B_n$ , and in the last equality we have used the space-homogeneous property of truncated symmetric  $\alpha$ -stable process.

Let  $(X_t)_{t \geq 0}$  be the truncated symmetric  $\alpha$ -stable process. By the Meyer construction for truncated  $\alpha$ -stable process (see [1, Remark 3.5]), there corresponds to a symmetric  $\alpha$ -stable process  $(X_t^*)_{t \geq 0}$  (on the same probability space), such that  $X_t = X_t^*$  for any  $t \in (0, N_1^*)$ , where

$$N_1^* = \inf \{t \geq 0 : |\Delta X_t^*| > 1\},$$

and  $\Delta X_t^* := X_t^* - X_{t-}^*$  denotes the jump of  $(X_t^*)_{t \geq 0}$  at time  $t$ . In the following, let

$$\tau_{B(0,r)}^* = \inf \{t > 0 : X_t^* \notin B(0, r)\}$$

be the first exit time from  $B(0, r)$  of the process  $(X_t^*)_{t \geq 0}$ . Note that, under  $\mathbb{P}^0$  the event  $\{X_t^* \in B(0, r), \forall t \in [0, 1]\}$  implies that the process  $(X_t^*)_{t \geq 0}$  does not have any jump bigger than 1. Then we find that there are constants  $C_4, \lambda_1^* > 0$  such that for all  $n \geq 1$  large enough,

$$\begin{aligned} \mathbb{P}^0(\tau_{B(0,r_n)} > 1) &\geq \mathbb{P}^0(\tau_{B(0,r_n)}^* > 1) \\ &= \mathbb{P}^0(\tau_{B(0,1)}^* > r_n^{-\alpha}) \\ &= \int_{B(0,1)} p_{B(0,1)}^*(r_n^{-\alpha}, 0, z) dz \\ &\geq C_4 e^{-\lambda_1^* r_n^{-\alpha}}. \end{aligned}$$

Here in the first equality we have used the scaling property of symmetric  $\alpha$ -stable process, in the second equality  $p_{B(0,1)}^*(t, x, y)$  denotes the Dirichlet heat kernel of the process  $(X_t^*)_{t \geq 0}$  on  $B(0, 1)$ , and the last inequality follows from lower bound of  $p_{B(0,1)}^*(t, x, y)$  established in [8, Theorem 1.1(ii)]. Hence, for  $n$  large enough,

$$(4.51) \quad T_1^V(\mathbf{1}_{B(x_n, 1)})(x_n) \geq C_4 \exp\left(-\lambda_1 n^{k_0 - \frac{\alpha}{d}}\right).$$

According to (4.50) and (4.51) above, we know that for any constant  $C > 0$ , the following inequality

$$T_1^V(\mathbf{1}_{B(x, 1)})(x) \leq CT_1^V(\mathbf{1}_D)(x).$$

does not hold for  $x = x_n$  when  $n$  large enough. In particular, [15, Condition 1.3, p. 5027] is not satisfied, and so the semigroup  $(T_t^V)_{t \geq 0}$  is not intrinsically ultracontractive.

However, for every  $R \geq 2$  and  $n \geq 1$  with  $n^{k_0} \leq R \leq (n+1)^{k_0}$ ,

$$\begin{aligned} |\{x \in \mathbb{R}^d : x \in A, |x| \geq R\}| &\leq \sum_{m=n}^{\infty} |B(x_m, r_m)| = \sum_{m=n}^{\infty} r_m^d \\ &= \sum_{m=n}^{\infty} m^{-\frac{dk_0}{\alpha} + 1} \leq C_5 n^{-\frac{dk_0}{\alpha} + 2} \\ &\leq C_6 \left((n+1)^{k_0}\right)^{-\frac{d}{\alpha} + \frac{2}{k_0}} \leq \frac{C_6}{R^{\frac{d}{\alpha} - \varepsilon}} \end{aligned}$$

holds for some constant  $C_6$  independent of  $R$ , where in the last inequality we have used the fact that  $\frac{2}{k_0} < \varepsilon$ . Therefore (1.13) holds true. By now we have finished the proof.  $\square$



## 5. APPENDIX

In this appendix, we will present the proofs of Propositions 1.1 and 1.2.

*Proof of Proposition 1.1.* Let  $(T_t)_{t \geq 0}$  be the Markov semigroup associated with the regular Dirichlet form  $(D, \mathcal{D}(D))$ . Under assumption **(A1)**, for every  $t > 0$ ,

$$\|T_t\|_{L^1(\mathbb{R}^d; dx) \rightarrow L^\infty(\mathbb{R}^d; dx)} = \sup_{x, y \in \mathbb{R}^d} p(t, x, y) \leq c_t.$$

According to [19, Theorem 3.3.15], the following super Poincaré inequality holds

$$\int f^2(x) dx \leq rD(f, f) + \beta(r) \left( \int |f(x)| dx \right)^2, \quad r > 0, f \in \mathcal{D}(D),$$

where

$$\beta(r) = \inf_{s \leq r, t > 0} \frac{s \|T_t\|_{L^1(\mathbb{R}^d; dx) \rightarrow L^\infty(\mathbb{R}^d; dx)}}{t} \exp\left(\frac{t}{s} - 1\right) \leq \|T_r\|_{L^1(\mathbb{R}^d; dx) \rightarrow L^\infty(\mathbb{R}^d; dx)} \leq c_r.$$

Therefore, we can take the reference symmetric function  $\mu$  in [21, (1.2)] to be Lebesgue measure.

Clearly, the potential function  $V$  satisfies [21, (1.3) and (1.5)]. Then, the desired assertion immediately follows from [21, Corollary 1.3].  $\square$

*Proof of Proposition 1.2.* For any  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,  $p^V(t, x, y) \leq p(t, x, y) \leq c_t$ , so the semigroup  $(T_t^V)_{t \geq 0}$  is ultracontractive. In particular, by the symmetric property of  $(T_t^V)_{t \geq 0}$  on  $L^2(\mathbb{R}^d; dx)$ , we know that  $\|T_t^V\|_{L^2(\mathbb{R}^d; dx) \rightarrow L^\infty(\mathbb{R}^d; dx)} < \infty$ . This, along with  $T_t^V \phi_1 = e^{-\lambda_1 t} \phi_1$  and  $\phi_1 \in L^2(\mathbb{R}^d; dx)$ , yields that there is a version of  $\phi_1$  which is bounded.

For any  $R > 0$ , let  $\phi_1^R(x) := e^{-\lambda_1 t} \int_{\{|y| \leq R\}} p^V(t, x, y) \phi_1(y) dy$ . For any  $y \in \mathbb{R}^d$  and  $t > 0$ , the function  $x \mapsto p_t^V(x, y)$  is continuous and  $p^V(t, x, y) \leq p(t, x, y) \leq c_t$ . According to the fact that  $\phi_1$  is locally  $L^1(\mathbb{R}^d; dx)$ -integrable and the dominated convergence theorem,  $\phi_1^R$  is also a continuous function. Now, for every fixed  $x_0 \in \mathbb{R}^d$ , let  $\{x_n\}_{n=1}^\infty \subseteq \mathbb{R}^d$  be a sequence such that  $\lim_{n \rightarrow \infty} x_n = x_0$ . Then we obtain

$$\begin{aligned} & |\phi_1(x_n) - \phi_1(x_0)| \\ &= e^{-\lambda_1 t} |T_t^V \phi_1(x_n) - T_t^V \phi_1(x_0)| \\ &= e^{-\lambda_1 t} \left| \int_{\mathbb{R}^d} p^V(t, x_n, y) \phi_1(y) dy - \int_{\mathbb{R}^d} p^V(t, x_0, y) \phi_1(y) dy \right| \\ &\leq |\phi_1^R(x_n) - \phi_1^R(x_0)| + 2 \sup_{x \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} (p^V(t, x, y))^2 dy \right)^{1/2} \left( \int_{\{|y| > R\}} \phi_1^2(y) dy \right)^{1/2} \\ &\leq |\phi_1^R(x_n) - \phi_1^R(x_0)| + 2\sqrt{c_t} \left( \int_{\{|y| > R\}} \phi_1^2(y) dy \right)^{1/2}. \end{aligned}$$

Letting  $n \rightarrow \infty$  and then  $R \rightarrow \infty$ , we arrive at  $\lim_{n \rightarrow \infty} \phi_1(x_n) = \phi_1(x)$ . Therefore there exists a version of  $\phi_1$  which is continuous.

Let  $(D^V, \mathcal{D}(D^V))$  be the Dirichlet form associated with  $(T_t^V)_{t \geq 0}$ . Due to the following variational principle

$$\lambda_1 = \inf \left\{ \frac{D^V(f, f)}{\int_{\mathbb{R}^d} f^2(x) dx} : f \in \mathcal{D}(D^V), f \neq 0 \right\} = D^V(\phi_1, \phi_1),$$

and the fact  $D^V(|\phi_1|, |\phi_1|) \leq D^V(\phi_1, \phi_1)$ , we know that  $\phi_1 \geq 0$ . Now, assume that  $\phi_1(x_0) = 0$  for some  $x_0 \in \mathbb{R}^d$ . Since  $p^V(t, x, y) > 0$  for any  $t > 0$  and  $x, y \in \mathbb{R}^d$ , and

$$\phi_1(x_0) = e^{-\lambda_1 t} \int_{\mathbb{R}^d} p^V(t, x_0, y) \phi_1(y) dy = 0,$$

we find by the continuity of  $\phi_1$  that  $\phi_1(x) = 0$  for every  $x \in \mathbb{R}^d$ . This contradiction implies that  $\phi_1 > 0$  is positive everywhere. The proof is complete.  $\square$

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